Math Review

Economics 100A

Fall 2021

1 Functions

1.1 Univariate Functions

In upper-division economics, you will be expected to be comfortable with graphing linear functions, solving for their slopes, and plotting their intercepts. Now, we are going to quickly review how to solve for slopes and intercepts of univariate (single variable) functions. Recall that the general form for a linear graph is $y = mx + b$ where m is the slope and b is the intercept.

Function	Slope	y-intercept	x-intercept
$y = 2x$	2	0	0
$y = -2x + 4$	-2	4	2
$x + y = 5$	-1	5	5
$3x + 4y = 10$	$-3/4$	$10/4$	$10/3$
$y/2 = 3 - x$	-2	6	3

These functions are all pretty straightforward, so be sure you can solve for their properties. As well, make sure you can graph nonlinear functions. Some popular examples that show up in this class include:

1. $y = \sqrt{x}$
2. $y = x^2$
3. $y = ln(x)$

This is not an exhaustive list, but you should definitely familiarize yourself with the shapes of these functions. It will come in handy later!

1.2 Multi-variate Functions

Since Math 10/20C is a prerequisite for this course, you are expected to have familiarity with multivariate calculus and functions. In this class, we will mainly work with functions that take on the form $z = f(x, y)$. In reality, you will not be expected to draw the three-dimensional graphs. However, just for reference, we can graph $z = \sqrt{xy}$. This graph would look something figure one.

Drawing these 3D graphs is incredibly difficult without graphing software, so we will take a different approach. Instead of graphing functions like this, we will create level curves for each function. This

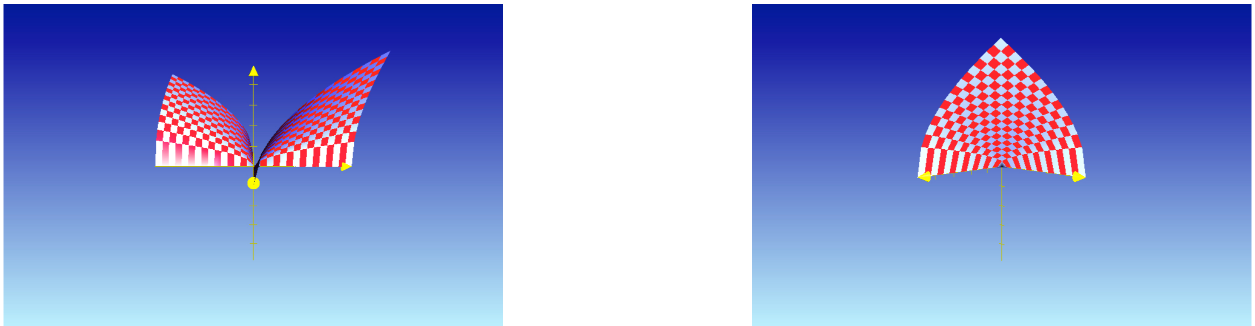


Figure 1: $z = \sqrt{xy}$

is what cartographers do to represent different three-dimensional terrain. Imagine taking our three-dimensional graph and slicing through a point, $z*$. This is how we get a single "level." I have added an image which shows how this slice works at $z = 10$. We can in fact do this for all levels across z, giving us the figure to the right.

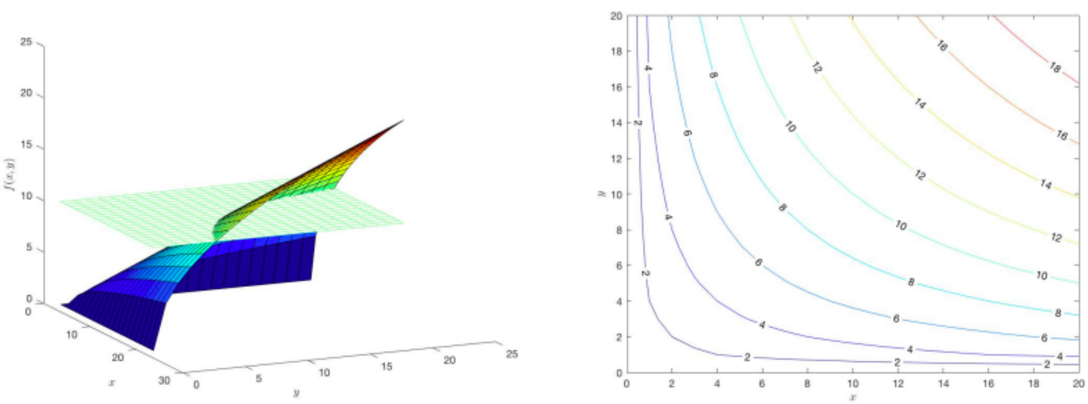


Figure 2: Level curve at $z = 10$.

2 Derivatives

2.1 Basic Rules

You should be comfortable with single-variable derivatives and their interpretations. Throughout this class, we will be differentiating multivariate functions. In case you need a brief reminder, the derivative of a function tells us the instantaneous rate of change, or the slope at a single point. Most derivatives we do in this class can be done using the power rule, but in some cases you will need to use the chain rule. In any event, we can go over what those look like.

2.1.1 Rules

1. If $f(x) = x^\alpha$ then $f'(x) = \alpha x^{\alpha-1}$. This only applies when α is constant.
2. $\frac{d}{dx} ln(x) = \frac{1}{x}$.
3. $\frac{d}{dx} ln[f(x)] = \frac{1}{f(x)} f'(x)$
4. $\frac{d}{dx} c = 0$, where c is a constant.
5. $\frac{d}{dx} \; c \cdot f(x) = c \cdot f'(x)$
6. If $f(x) = u(x) + v(x)$, then $f'(x) = v'(x) + u'(x)$
7. If $f(x) = u(v(x))$, then $f'(x) = \frac{du(v)}{dv} \cdot \frac{dv(x)}{dx} = u'(v(x)) \cdot v'(x)$
8. If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x)$
9. If $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{v(x)^2}$

2.2 Partial Derivatives

Partial derivatives are derivatives of multivariate functions with respect to a single variable. We treat each other variable as a constant.

Examples

Function	Derivative of x	Derivative of y
$f(x) = x^2$	$2x$	0
$f(x) = x$	1	0
$f(x, y) = xy$	y	x
$f(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$	$\frac{1}{3} x^{\frac{-2}{3}} y^{\frac{2}{3}}$	$\frac{2}{3} y^{\frac{-1}{3}} x^{\frac{1}{3}}$
$f(x, y) = \log(x^{\frac{1}{3}} y^{\frac{2}{3}})$	$\frac{1}{3x}$	$\frac{2}{3y}$
$f(x, y) = \log(x^{\frac{1}{3}} y^{\frac{2}{3}}) + e^{3x}$	$\frac{1}{3x} + 3e^{3x}$	$\frac{2}{3y}$
$f(x, y) = \log(x) + y$	$\frac{1}{x}$	1

There is no royal road to math. If you feel uncomfortable with any of these derivatives, review and practice. Practice is the only way you actually improve in math; it is not a spectator sport.

3 Optimization

3.1 Constrained Optimization

Economics is concerned with efficiency: how do we efficiently allocate our scarce goods? Think about your own personal experiences going to the store. You typically have some budget which allows you to purchase a certain number of goods. Following your budget constraint, you buy whatever bundle of goods satisfies your needs and/or desires. In economic terms, you are maximizing utility, subject to some budget constraint. We will go into what exactly that means in a few sections, but for now just think about maximizing happiness subject to a certain constraint: money, time, and so on.

We need to identify three things:

1. *Objective Function*: What function are we maximizing?
2. *Constraint*: What are we limited to?
3. *Choice Variables*: What variables will we maximize?

In terms of math, we write

$$\begin{gathered} max f(x,y) \\ \text{s.t.} \quad h(x,y) \geq 0 \end{gathered}$$

What we are doing here is maximizing (though we could minimize, but we would have to write $\min(f(x,y))$ some function subject to a constraint. In this case, our objective function is $f(x,y)$ and our constraint is $h(x,y)$. Our choice variables are x and y in this case, and in most cases for this class.

If you are interested in the math behind the constraint: we are restricting the number of sets that are feasible. We write this as an inequality. However, in our class, we will typically write our constraint as an equality. This is more for economic reasons, as you will soon see.

3.1.1 Substitution Method

1. Rewrite the constraint in terms of x or y.
2. Substitute the constraint into the original function for either x or y.
3. Take the first order condition with respect to the single variable in the objective function.
4. Rearrange and solve for the choice variable. Then, use the constraint to solve for the other choice variable.

Example Say we have $f(x,y) = xy$ s.t. $x + y = 10$. Step one tells us to rewrite the constraint in terms of a single variable, so we have

$$y = 10 - x$$
$$f(x,y) = x(10-x)$$
$$f(x,y) = 10x - x^2$$
$$\frac{d}{dx}(10x - x^2) = 10 - 2x$$
$$2x = 10$$
$$x = 5$$

Substitute our new x into the constraint to get $y = 10 - 5$ or $y = 5$.

Example Two Let's make things just a tad more difficult. Say we have to maximize x^2y s.t. $c - x - 2y = 0$. In this case we are not given a number for c, so we will solve for a general case.

$$y = \frac{c}{2} - \frac{x}{2}$$

Substitute the constraint into our objective function: $x^2(\frac{c}{2} - \frac{x}{2}) = \frac{cx^2}{2} - \frac{x^3}{2}$

Take the first order condition: $\frac{d}{dx} = cx - \frac{3}{2}x^2 = 0$

We can factor out an x to get $x(c - \frac{3x}{2})$

As you can see, x will either be 0 or $\frac{2c}{3}$. In this case, does 0 make sense for maximization? No, so the answer is $\frac{2c}{3}$. If we plug in to solve for y, we get: $\frac{c}{2} - \frac{c}{3} = \frac{c}{6}$.

Et voila. There we have our constrained optimization.

3.1.2 Lagrangian

The Lagrange multiplier method for solving constrained optimization is very powerful but more time consuming than the substitution method. Most students prefer using the substitution method, but it is good to be familiar with both. In any event, these notes cover how you would use a Lagrange multiplier.

1. Set up $\mathcal{L} = Objective + \lambda[constraint] = f(x,y) + \lambda[h(x,y)]$
   (a) Set your constraint equal to zero so that you have $c - x - y = 0$ inside the constraint
2. Take three first order conditions: one with respect to x, one with respect to y, and one with respect to λ.
3. Solve $\frac{df}{dx}$ and $\frac{df}{dy}$ in terms of λ and then equate the two.
4. Rewrite the above equation in terms of either x or y.
5. Substitute x or y into the budget constraint (i.e. the partial with respect to λ).

6. Finally, solve for x or y and use the budget constraint to solve for the other choice variable.

Example Let's go back to x^2y s.t. $c - x - 2y = 0$. Set up the Lagrange as follows: $\mathcal{L} = x^2y + \lambda[c - x - 2y]$

$$\frac{d\mathcal{L}}{dx} = 2xy - \lambda = 0.$$

$$\frac{d\mathcal{L}}{dy} = x^2 - 2\lambda = 0.$$

$$\frac{d\mathcal{L}}{d\lambda} = c - x - 2y = 0.$$

Rewrite both x and y in terms of λ and equate them.

$$\tfrac{1}{2}x^2 = 2xy$$

$$x = 4y$$

Substitute into the constraint

$$c - 4y - 2y = 0$$

$$c - 6y = 0$$

$$y = \tfrac{1}{6}c$$

Now that we have y, we can solve for x.

$$c - x - 2y = 0$$

$$c - x - 2(\tfrac{1}{6}c) = 0$$

$$x = c - \tfrac{2}{6}c$$

$$x = \tfrac{4}{6}c$$

Elasticity

Economics 100A

Fall 2021

1 Elasticity

1.1 Elasticity Overview

Elasticity is a measure of responsiveness. More specifically, we measure the change in variable due to the change of another variable. For example, if the price of a good changes, how does the quantity demanded change? Since we know calculus, we can measure this quite easily. Assume that we have an independent and a dependent good. Our set up will begin as follows:

$$y = f(x) \tag{1}$$

If we change x, how does y change in response? Let's start with a very simple case. Say $x_1 = 5$ and decreases to $x_1 = 2$. Accordingly, y_1 starts at 4 but increases to 6. We can calculate the change in each variable as the ratio of the changes:

$$\frac{y_2 - y_1}{x_2 - x_2} = \frac{\Delta y}{\Delta x} \tag{2}$$

In the example above, we would write $\frac{6-2}{2-5}$ which is just $\frac{4}{-3}$. Therefore, we see that we have a negative relationship between the two variables.

As you may remember from your calculus classes, $\frac{\Delta y}{\Delta x}$ is just the slope of a function at point x. In other words, it is a derivative. So, we can see that initially, things are as simple as a derivative. However, imagine the following scenario: the price of a good increases by \$100 and quantity falls by 1000 units. Is this a large change? Well, for a car, the price change is not much. For a meal, however, the change is steep. We need a better way of finding the responsiveness of variables. Currently, we are using absolute change. Can you think of a better way of representing change?

If you thought about converting absolute change into percentage change, you are correct. Let us think back to our initial, simpler problem, where x_1 changes from 5 to 2 and y_1 increases from 4 to 6. To measure the percentage changes in each of these variables, we need to rewrite our fraction to be:

$$\epsilon = \frac{\dfrac{y_2 - y_1}{y_1}}{\dfrac{x_2 - x_1}{x_1}} = \frac{\%\Delta y}{\%\Delta x} \tag{3}$$

Now, this is what would be called a continuous equation for elasticity. This is because we are measuring the elasticity between two different points. But what if you are not given two different points? What if you need to solve for the elasticity at a discrete point?

If you are thinking about taking a derivative and converting it into a percent, you are right again. Let's write this out in Leibniz's notation.

$$\epsilon_{y,x} = \frac{d_y}{d_x}\frac{x}{y} \tag{4}$$

Note 1. For those interested in the derivation of this formula, consider the continuous equation $\frac{\%\Delta y}{\%\Delta x}$. Let's rewrite it so that we get $\frac{\Delta y}{\Delta x}\frac{x}{y}$. Now, if Δx and Δy are both very small, Δx and Δy becomes $\frac{d_y}{d_x}$ and we are left with $\dfrac{d_y}{d_x}\dfrac{x}{y}$.

And there we have it. We were able to derive two very important equations in Economics. I should note here that the specific variables for x and y can be anything. Remember: elasticity is a measure of responsiveness. Some famous examples of elasticity include

1. **Income Elasticity of Demand**: This is a measure of percent change in quantity demanded in response to a 1% change in consumer income.
2. **Price Elasticity of Demand**: This is a measure of percent change in quantity demanded in response to a 1% change in price.
3. **Cross-Price Elasticity of Demand**: This is a measure of percent change in quantity demanded for one good in response to a 1% change in price of another good.

Try to write those elasticity formulas based on the notation we have used hitherto. Solutions will be at the bottom of the page.

1.2 Practice Problems

1.2.1 Basic Problems[1]

1. If demand is represented by $Q = 10 - 2P$, calculate the price elasticity.
2. Elasticity of supply is $\frac{3}{4}$, and a price change causes q to decrease by 9%. What is the percentage price change?
3. Suppose $Q = 10000 - 10P$. P is initially 500 but decreases to 400, what is the price elasticity of demand?
   (a) Now suppose the current market price is \$400. A firm asks you, the economic expert, whether or not they should increase the price. What do you say, and how do you justify your answer?

[1] By "basic" I do not mean trivial. It is fine if you struggle with these at first. These types of problems are typically just calculations and do not test your conceptual understanding.

1.2.2 Medium Problems

1. Demand for good x: $Q_x = 10 - 2P_x + 3P_y$. Calculate the cross-price elasticity.

2. Elasticity of demand for a consumer's income is $\frac{-1}{2}$. A consumer's income is initially $\$40,000$/year and they buy 50 of good x each year. How much of good x will they buy if income rises to $\$56,000$ per year?

1.2.3 Hard Problems

1. How does adding a positive constant linear term to $f(x)$ affect elasticity? Assume f is strictly positive and increasing.

2. How does taking the inverse of a function affect its elasticity?

1.3 Solutions

1.3.1 Basic Problems

1. $\epsilon_{Q,P} = \frac{d_Q}{d_P}\frac{P}{Q} = (-2)\frac{P}{Q}$

   $= (-2)\frac{P}{10-2P}$

   $= \frac{-2P}{10-2P}$

2. $\epsilon_{Q,P} = \frac{\Delta\%Q}{\Delta\%P}$

   $\frac{3}{4} = \frac{-9\%}{\Delta\%P}$

   $\Delta\%P = (-9\%)\frac{4}{3}$

   $\Delta\%P = -12\%$

3. $P_1 = 500, P_2 = 400. Q_1 = 10,000 - 10(500) = 5000. Q_2 = 10,000 = 10(400) = 6000.$

   $\epsilon = \frac{\frac{6000-5000}{5000}}{\frac{400-500}{500}}$

   $= \frac{.2}{-.2}$

$= 1$

(a) Since it is unit elastic, I would recommend not changing the price at all. At this point, the percentage change in price would equal a percentage change in quantity. If we had an inelastic good, for example, a change in price would be bigger than a change in quantity demanded.

1.3.2 Medium Problems

1. $\epsilon_{P_y} = \dfrac{dQ_x}{dP_y} \text{ x } \dfrac{P_y}{Q_x}$

   $= (3)\dfrac{P_y}{10 - P_x + 3P_y}$

   $= \dfrac{3P_y}{10 - P_x + 3P_y}$

2. $\epsilon_{Q,I} = \dfrac{\%\Delta Q}{\%\Delta P}$

   $= \%\Delta I = \dfrac{56 - 40}{40} = \dfrac{16}{40} = .4$

   $= \%\Delta Q = \epsilon \text{ x } \%\Delta I = \dfrac{-1}{2} \text{ x } \dfrac{2}{5} = \dfrac{-1}{5}$

   $\dfrac{Q_2 - 50}{50} = \dfrac{-1}{5}$

   $Q_2 - 50 = -10$

   $Q_2 = 40$

1.3.3 Hard Problems

1. $\epsilon_{f(x)} = \dfrac{dy}{dx} \text{ x } \dfrac{x}{y}$

   We need to create a new function: $g(x) = f(x) + c$

   $\epsilon_{g(x)} = \dfrac{dy}{dx} \text{ x } \dfrac{x}{y + c}$

   $\therefore \epsilon_{f(x)} > \epsilon_{g(x)}$

2. The inverse of $f(x)$ is $\frac{1}{f(x)}$. Let's solve for a specific function before generalizing our answer.

   $f(x) = x^2$ so the inverse is $f^-1(x) = x^{-2}$.

$\epsilon = \dfrac{df}{dx}\ \dfrac{x}{f} = (2x)\ \dfrac{x}{x^2}$

$\epsilon = \dfrac{2x^2}{x^2} = 2$

Now for the inverse: $\epsilon = \dfrac{df}{dx}\ \dfrac{x}{f} = (-2x^{-3})\ \dfrac{x}{x^{-2}}$

$\epsilon = \dfrac{-\ 2x^{-2}}{x^{-2}} = -2$

We see that taking the inverse of a function keeps the value the same but changes its sign. A general form for this would go as follows:

$\epsilon_{y,x} = \dfrac{dy}{dx}\ \dfrac{x}{y}.\ g = \frac{1}{y}.$

$\epsilon_{g,x} = \dfrac{dg}{dx}\ \dfrac{x}{g}$

$\epsilon_{g,x} = \dfrac{1}{-y^2}\ (g)\ \dfrac{x}{\frac{1}{y}}$

$-g\frac{x}{y} = -\epsilon_{y,x}$

Preferences

Economics 100A
Fall 2021

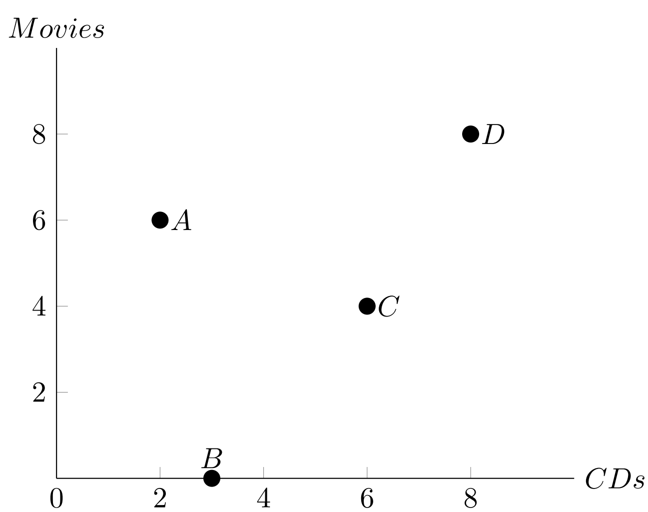
1 Preferences

1.1 Introduction

Econ 100A is all about the consumer side of the market. We will slowly build our intuition for understanding the demand curve. Before we get to that point, however, we should build our understanding of how consumers actually rank different goods. Let's say you go to the grocery store: what determines the ratio in which you purchase apples and oranges? If you have a strong preference for oranges, maybe you consume ten oranges for every one apple. If you are indifferent between the two, maybe you consume them based on whatever is on sale. These are real-life examples we can attempt to model in this class.

Put another way, consumers have certain preferences that determine what they consume. If I gave everyone who attended SI $10 to purchase dinner, everyone would come back with something different. In this example, the difference between what we purchase has to do with our **preferences**.

In the real world, we have to choose between a lot of different goods. In this class, we will keep things simple: we will study binary choices. All this means is that we will take two goods and see how consumers rank different bundles of each good. Consider the following: consumers are asked to rank how many movies and CDs they would like to purchase. Four consumers provide the following answers:



Can you see which person likes CDs much more than movies? Can you see who is indifferent between the two? Who likes CDs slightly more than movies? Think about this based on how each consumer ranked the two goods.

It seems like consumer B has a strong preference for CDs, as they do not want to consume any movies. Conversely, consumer D is indifferent between the two, opting to consume the maximum of each good. Consumer C seems to slightly prefer CDs over movies, but not by a wide margin. Finally, consumer A prefers movies by a pretty significant margin.

1.2 Notation and Properties

Now that we have an understanding of what exactly we want to model, we can start with the math behind preferences. Say we have goods x and y. If a consumer likes x just as much as y, we write $x \succsim y$. If a consumer is indifferent between x and y, we write $x \backsim y$. If a consumer strictly prefers x to y, we write $x \succ y$. To summarize:

1. $\succsim$: Weakly preferred to ("liked at least as much as")
2. $\backsim$: Indifferent, when $x \succsim y$ <u>and</u> $y \succsim x$
3. $\succ$: Strictly preferred to, when $x \succsim y$ but <u>not</u> $y \succsim x$.

To make this a useful tool, we should institute some basic assumptions. I should probably mention here that these assumptions are **not** realistic or empirically grounded. People routinely violate these assumptions. The first set of assumptions are called **rational preferences**, and the second set is called **well-behaved**. Before I outline them, let's assume the two goods x and y are consumed together in a bundles A and B.

1. <u>**Complete**</u>: for any two bundles A and B, it must be the case that $A \succsim B$ or $B \succsim A$, or both: $A \backsim B$.
2. <u>**Reflexive**</u>: any bundle is at least as good as itself: $A \succsim A$. This is a trivial assumption that is not often mentioned explicitly.
3. <u>**Transitive**</u>: For bundles A, B, C, if $A \succ B$ and $B \succ C$, then $A \succ C$. This essentially allows us to make predictable, unambiguous choices about bundles.

These assumptions fall into the category of **rational preferences**. These are the bare minimum that all preferences must have. The second variety of assumptions are called **well-behaved**. There are two assumptions here:

1. <u>**Monotonicity**</u>: A consumer would always prefer having more of a good. Their preferences never satiate. Expressed formally, for any two bundles $A = (x_1, x_2)$ and $B = (y_1, y_2)$, if $x_1 \geq y_1$, $x_2 \geq y_2$, and $x_i > y_i$ for $i = 1$ or 2, then $A \succ B$.
2. <u>**Convexity**</u>: A consumer prefers to consume averages rather than extremes. Expressed formally: if $x \succsim y$, and $y \succsim z$, then $\lambda x + (1 - \lambda) y \succsim z$ for any $\lambda \in [0, 1]$.

The first three assumptions are necessary for doing basic consumer analysis, but the second two make our lives much easier. Before advancing, let's test our understanding of preferences.

1. From the bundles A, B, C, I have $A \succ B$, $B \succ C$, and $C \succ A$. Are my preferences transitive? Are they complete?
2. I purchase fruit based on their sizes. I prefer bigger fruit. Are my preferences rational?
3. When I have milk tea, I put in a single spoonful of sugar. Any more and I feel sick. Is this preference monotonic?
4. Lucy faces two ways of spending her evenings: violin lessons or walks on the beach. She thinks violin lessons would only be worth the effort if she really committed to taking a lot a week; put another way, she would rather take none than only a few. Are her preferences convex?

Answers below:

1. My preferences are not transitive, because $A \succ B \succ C \succ A$. They are complete because I ranked all bundles.
2. The preferences are indeed rational. I can rank all of the fruits in order from most preferred (the biggest) to least preferred (the smallest), satisfying the completeness and transitivity conditions.
3. The preference is not monotonic because I have a satiation point at one spoonful of sugar. Remember: monotonic preferences state that a consumer is always better off by consuming more.
4. Her preferences are concave. She does not prefer averages of the two goods; she would either fully commit to violin or not commit at all.

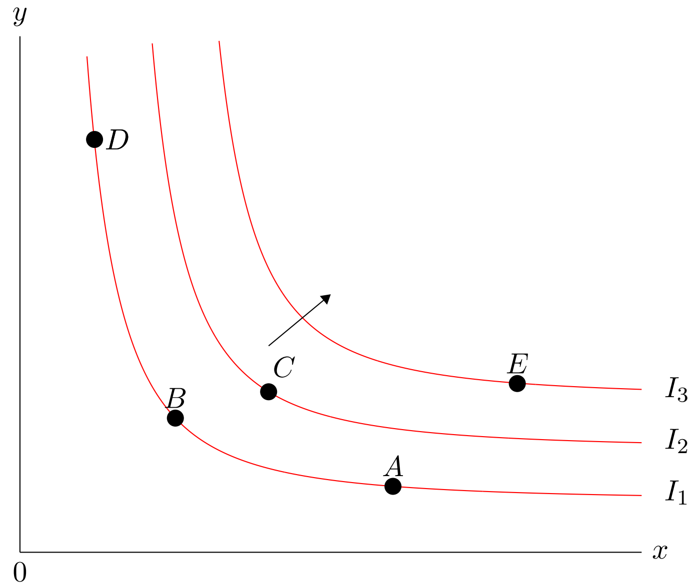
1.3 Indifference Curves

These assumptions allow us to graph consumer preferences using **indifference curves**. An indifference curve is just a graphical representation of a consumer's preferences. Each point on an indifference curve represents a bundle. Indifference curves indicate the set of bundles that the consumer feels indifferent between.

In figure one, we see three different indifference curves. Any point on each curve represents a point of indifference. For example, on I_1, $A \backsim B \backsim D$. However, due to our monotonicity assumptions, we prefer to consume more. So, $I_2 \succ I_1$, and likewise $C \succ B$. It is useful to draw arrows indicating which direction leads to a higher curve. In the absence of an arrow, assume that the farthest northeast point is the best point, or the one that the consumer prefers.

The figure also makes it easy to see how the transitive property works. Point E is on a higher indifference curve than point C. So, $E \succ C$. However, we also know that $C \succ B$. As we can see from the graph, $E \succ B$ as well.

Figure 1: Indifference Curves



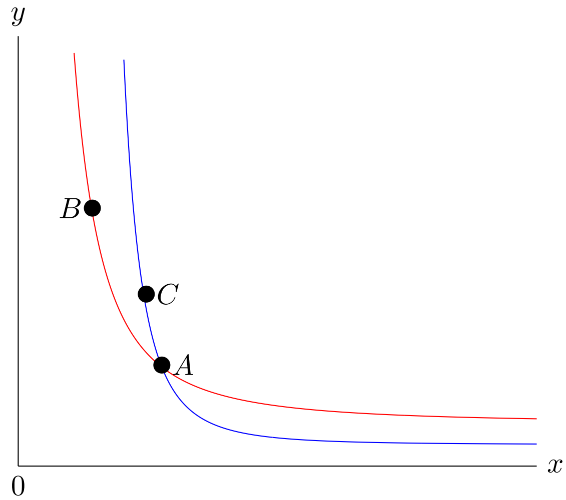
To summarize:

1. For any point **above** the indifference curve, the consumer strictly prefers it to any point **on** the indifference curve.
2. For any point **on** the indifference curve, the consumer is indifferent between it and any point **on** the indifference curve.
3. For any point **below** the indifference curve, the consumer is strictly does not prefer it and any point **on** the indifference curve.

There are a few more points about indifference curves we should think about. They are as follows:

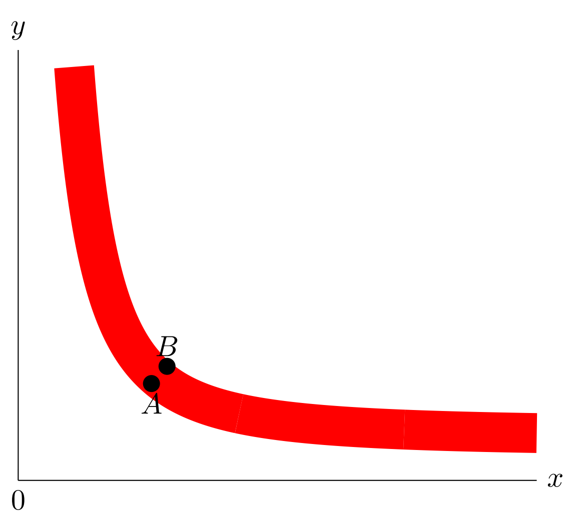
1. **Higher indifference curves are more preferred**: This is a direct result of our monotonicity assumption. We typically draw arrows indicating in which direction indifference curves move. The easiest way to think about this assumption is that people always prefer having a bundle with more goods than one with fewer goods.
2. **Every point has an indifference curve going through it**: This is a result of our completeness assumption. For obvious reasons, we do not draw an infinite number of indifference curves; we draw a few just to get a rough idea of the shape of the preferences. However, since the completeness assumption states that our consumer can rank bundles, which allows us to represent their preferences graphically or algebraically.
3. **Indifference curves never cross**: Imagine crossing indifference curves in your head. Why might this not be allowed? Well, as you can see below, if we had crossing indifference curves, we would have that $A \backsim B \backsim C$. However, we can clearly see that $C \succ B$ because it is on a

higher indifference curve! We now have that $C \succ B \backsim A \backsim C$. This is a clear violation of our rational preferences, and therefore we should never see crossing indifference curves.



4. **Well-behaved indifference curves are downward sloping**: This comes from our monotonicity assumption. Let's say that you can consume both milk tea and coffee. If I feel generous and give you a cup of coffee, you would be on a higher indifference curve, since your bundle now has more goods. Now, what if I wasn't actually feeling that generous, and I wanted to make you as well off as before. Now, if I gave you coffee, I would have to take away one of your milk teas. I would take away just enough tea to make you as content as before. Let's say x_1 is coffee and x_2 is tea. By giving you coffee and taking away tea, we are increasing x_1 but decreasing x_2, which is the exact definition of a negative slope.

5. **Indifference curves are thin**: This also comes from monotonicity. If you had a thick curve, you could have two points: one above another. If one point represented a larger bundle, yet was on the same indifference curve, it would violate our monotonicity assumption. We can see this in the figure below, as bundle $B \succ A$ despite both being on the same indifference curve, implying $B \backsim A$.



6. **Well-behaved indifference curves are convex**: This comes from the convexity assumption, as the name suggests. This just says that our indifference curves bow in toward the origin, due to the fact that consumers prefer a weighted average of both goods rather than consuming an extreme amount of one good. Indifference curves that curve outward imply irrational preferences.

1.4 Indifference Curves in action

In this section, we are going to think about different indifference curves to suit our preferences. Before we proceed, let's just bear in mind the necessary properties of indifference curves:

- Any point on the same indifference curve must be equally preferred by the consumer
- Higher indifference curves are strictly preferred to lower indifference curves
- We draw arrows indicating the direction in which indifference curves move
- The slope of an indifference curve tells us how consumers substitute each good to retain indifference.
  1. A negative slope means that if I give you a good x_1, I must take take away some amount of your other good x_2 to make you as well off as before. This means both items are economics goods.
  2. A positive slope means that if I give you more of a good x_1, I must also give you more of x_2 to make you indifferent to before. This indicates that x_1 is an economic bad.
  3. A slope of 0 (horizontal line) means that no matter how much additional x_1 you receive, you remain indifferent to before.
  4. A slope of ∞ (vertical line) means that no matter how much additional x_2 you receive, you remain indifferent to before.

With all of that in mind, let's think about a few examples...

(a) Let's start with a simple example. Suppose we have someone who spends part of their income on apples and part on oranges. Assuming that they have well-behaved and rational preferences, their indifference curves should look something like graph (a). This is a standard set of indifference curves which satisfy all of our properties: they slope downward, are convex, and do not cross.

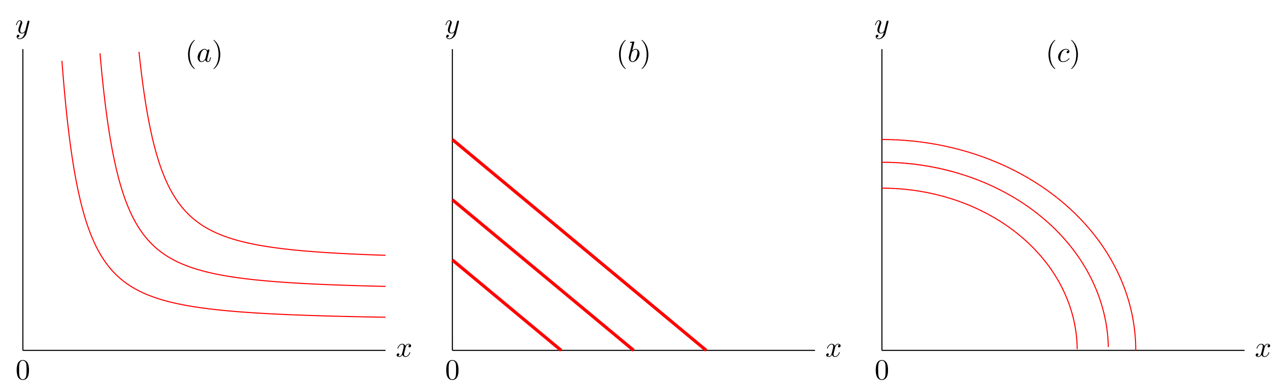
(b) This consumer has indifference curves with a constant slope. We see that they slope downward, but they are not *strictly* convex. We call these indifference curves **perfect substitutes**, because the consumer is willing to substitute goods at a constant rate. Monotonicity is satisfied as well, as the downward-sloping curves move northeast. An example of perfect substitutes in real life could be dimes and nickels: I am always willing to substitute two nickels for one dime, because they have the exact same value.

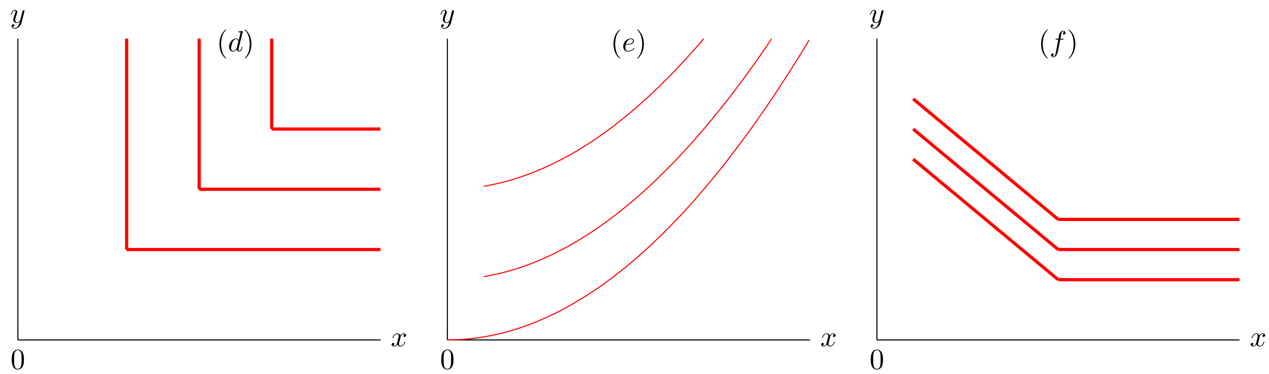
(c) This consumer prefers consuming extreme bundles. The curves are monotonic and downward-sloping, but clearly they are not convex. This is like Lucy's preference for violin lessons and walks on the beach: she would rather have an extreme number of one rather than a mix of both.

(d) These L-shaped indifference curves are technically convex, though they are clearly monotonic. These indifference curves are called **perfect complements**. The point at which the curves seem to bend is call a "kink point." Notice how if you increase x_1 or x_2 in either direction, you remain totally indifferent. Think about these in terms of right shoes and left shoes. If you have three pairs of shoes (so three right shoes and three left shoes), and I give you 100 right shoes, you remain as well off as before. That's kind of a silly example, but the point is that you have to consume each good together. The goods do not need to be consumed in on-to-one ratios, either. Another classic example of perfect complements could be two tires for one bicycle. The key takeaway is that to get to a higher indifference curve, it is not enough to receive more of only one of the goods. The consumer must receive more of both good.

(e) These indifference curves have a positive slope, indicating that one of the goods is actually a "bad." How do we know this? As we give the consumer more x_1, we have to also give them more x_2 in order to make them as well off as before!

(f) These indifference curves are a fun mix of (b) and (d)! We see that there is a kink point: to the left, we have a horizontal line like perfect complements; to the left, we have what appear to be perfect substitutes. These indifference curves are hard to come by in real life, but an example could come from a grading rubric. Say your professor determines your grade as a weighted average of two exams or, if it is higher, only one exam. This would mean that there are a number of different situations in which a student would be indifferent: for example, if their x_1 was 60 and their x_2 was 60, the average of the two are the exact same as x_1, leaving the student indifferent (if this goes over your head, don't worry. We will cover it further in SI).





2 Utility

2.1 A brief note on economic models

This section won't be covered in class, so feel free to skip it if you want. I do think, though, that it is at least responsible to comment on economic models, and how controversial their efficacies are. In the models we study for class, there ends up being a lot of ambiguity about whether the model is being chosen because it actually captures some kind of genuine relationship, or because it has certain desirable formal features.

Most economic models use exogenous variables (i.e., variables outside of the model) to explain endogenous variables (variables within the model). In this class, for example, we will see how consumers react to changes in prices and changes in income. These are all typically given to us, so they are exogenous; the endogenous variables are determined using each model.

The main thing to be aware of in this class is that these models are highly simplified and rely on a number of potentially unrealistic assumptions about consumer behavior. Sometimes they tell us interesting things about the world, but it is always important to be mindful of the shortcomings of these models.

2.2 Representing Utility

Until this point, we have been talking a lot about indifference curves and preferences. We know that we can represent certain preferences using indifference curves, but what if we wanted to be more analytical about ranking preferences? Since preferences are not a function, we have to get creative about representing preferences. We need some way of assigning a number to a certain bundle of goods. We do not really care about the number by itself; this number only has meaning in the context of other numbers generated from different bundles of goods.

In comes the **utility function**. Utility functions are formulas that assign numbers to a bundle of different goods. It normally takes the form $u(x_1, x_2)$, where u is the consumer's utility from consuming goods x_1 and x_2. Now, u can be thought of the amount of happiness a consumer gets from consuming the bundle. If u is higher for one bundle A than for bundle B, we say that the consumer gets more utility from A, or that $A \succ B$. Likewise, if $u_1 = u_2$, then we say that the consumer is indifferent between the two bundles. Written formally, $A \succ B \Leftrightarrow U(A) \succ U(B)$. Likewise, $A \backsim B \Leftrightarrow U(A) = U(B)$.

The important takeaway here is that our utility functions only tell us whether a consumer prefers one bundle to another; it does not tell us the magnitude of the preference. This kind of ordering is **ordinal**, not cardinal. Only the *order* of the preferences matters. This leads to a powerful realization about monotonicity: we can make transformations to functions so long as we preserve the order. Let's think of an example: suppose we have $x \succsim y \succsim z$. By transitivity, $x \succsim z$. If we wanted to write this in terms of our utility functions, we would write $u(x) \succsim u(y) \succsim u(z)$. The important thing here is that the **order** is saved – not the values themselves. The values of utility are completely meaningless. If $u(x) = 3$, $u(y) = 2$, and $u(z) = 1$, the equality is satisfied. However, $u(x) = 10000$, $u(y) = 1000$ and $u(z) = 100$ could also satisfy the equation, despite the much wider differences in utilities.

A positive monotonic transformation is one that transforms the utility function but keeps the order the same. If we, for example, wanted to divide u by 100 to make the numbers a bit nicer, we could do that with no problem. The important thing is that we preserve the order of the preferences. In summary: preference relations do **not** have unique utility representations. They only need to have their orders represented.

2.3 Marginal rate of substitution

As it turns out, our indifference curves correspond nicely with our utility functions. This should not be surprising, but if you graph the utility function, you get the consumer's indifference curves (it would be a pretty bad model if they did not correspond). In any event, our indifference curves show us points of indifference for a certain consumer. This means that bundles with the same utility are on the same indifference curve. $u(x) = u(y)$ implies that $x \backsim y$ which implies that both bundles are on the same curve.

Look back to figure one for a moment. These three indifference curves have no utility labels, but we can clearly see that since $I_3 \succ I_2 \succ I_1$, $u_3 > u_2 > u_1$. Each curve represents a different level of utility.

How should we interpret the slope of the indifference curves? Turns out that the slope has a very specific technical name, **marginal rate of substitution**. I am sure you are familiar with the concept of marginal utility, or the utility from consuming an additional unit of good x. In this case, the MRS just tells us the number of good y the consumer will give up in exchange for one unit of good x. You can write this slope as

$$\frac{\Delta y}{\Delta x} \tag{1}$$

which just says that the change in y over the change in x is the slope. However, since Δy represents a very small change in y, we can actually rewrite the formula for MRS as:

$$\frac{\frac{\delta U(x,y)}{\delta x}}{\frac{\delta U(x,y)}{\delta y}} = -\frac{MU_x}{MU_y} \tag{2}$$

Notice how this formula is really just the ratio of marginal utilities. The marginal utility of good x captures how much my utility changes by consuming a little bit more of good x, holding everything else constant. Notice how the slope is negative. We have covered this a good amount, but it is important once again to understand this intuitively. As x increases by one unit, we must decrease y to stay on the same indifference curve. Therefore, we have a negative relationship!

One final note before getting into examples: MRS is not always constant; in fact, it is basically a function. It takes on different values depending on a consumer's willingness to substitute.

2.4 Examples

2.4.1 Cobb-Douglas

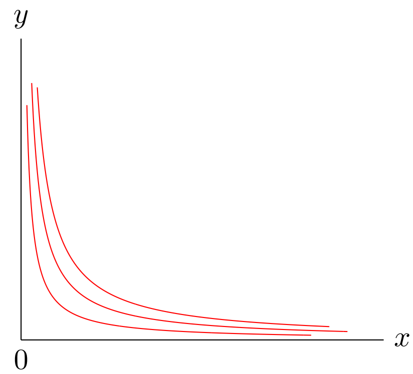
Cobb-Douglas utility functions take on the form

$$u(x,y) = x^\alpha y^\beta \tag{3}$$

These preferences have well-behaved, standard indifference curves. They never touch the axes. Typically, the exponents represent the share of the consumer's income allocated to each good, meaning that these preferences are convex: consumers prefer mixing. I will go through an example of how to find the MRS of Cobb-Douglas preferences right here:

$$u(x,y) = x^2y^3$$
$$MU_x = 2xy^3$$
$$MU_y = 3x^2y^2$$
$$MRS = \frac{MU_x}{MU_y} = \frac{2xy^3}{3x^2y^2}$$
$$MRS = \frac{MU_x}{MU_y} = \frac{2y}{3x}$$

And there you have it. That is how easy it is to find the marginal rate of substitution. These indifference curves are well-behaved because they are monotonic and convex. We know that they are convex because as x increases, y must decrease to remain on the same indifference curve.



2.4.2 Quasi-Linear

Easily the most confusing utility function, quasi-linear functions normally take on the form:

$$u(x, y) = f(x) + c \tag{4}$$

where the first term is a function and the second term is a constant. Popular examples include $u(x, y) = ln(x) + y$. These indifference curves are normally wider than Cobb-Douglas curves, but also only move in one direction: each indifference curve is just a parallel shift of other indifference curves. These preferences are pretty useful for modeling someone who is choosing between one necessity and one luxury good. We will prove this later. However, I can prove right now that the shifts are all parallel. Let's rewrite u such that

$$y = u - f(x)$$

We can see that each new value of u shifts the y-intercept, and the slope is exactly the same. Another way we can do this is by finding the MRS of the general case:

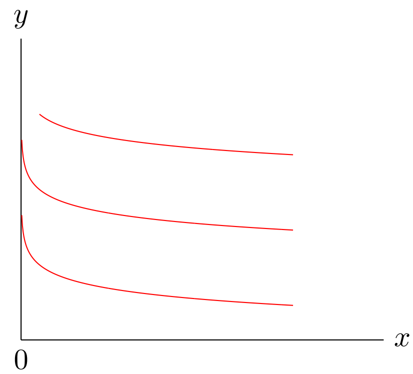
$$MRS = \frac{\delta u(x, y)/\delta x}{\delta u(x, y)/\delta y} = -f'(x)/1 = -f'(x)$$

As you can see, the MRS does not depend on y. Therefore, for the same level of x, the slope of the indifference curve will be the same regardless of whether we move y.

Let's do a more concrete example. Suppose $u(x, y) = ln(x) + y$. We can write out the MRS like so:

$$MRS = -\frac{MU_x}{MU_y} = -\frac{\frac{1}{x}}{1} = -\frac{1}{x}$$

Y is not in the picture at all, affirming once again that the slope is the same if we fix x.



2.4.3 Perfect Substitutes

Perfect substitutes take on the form

$$u(x, y) = \alpha x + \beta y \tag{5}$$

where a consumer is as happy to consume α units of x as they are β units of y. These indifference curves are linear and weakly convex. Let's think of an example: suppose I am indifferent between one dime or two nickels. This would mean that $\alpha = 2$ and $\beta = 1$, giving us the utility function $u(x, y) = 2x + y$.

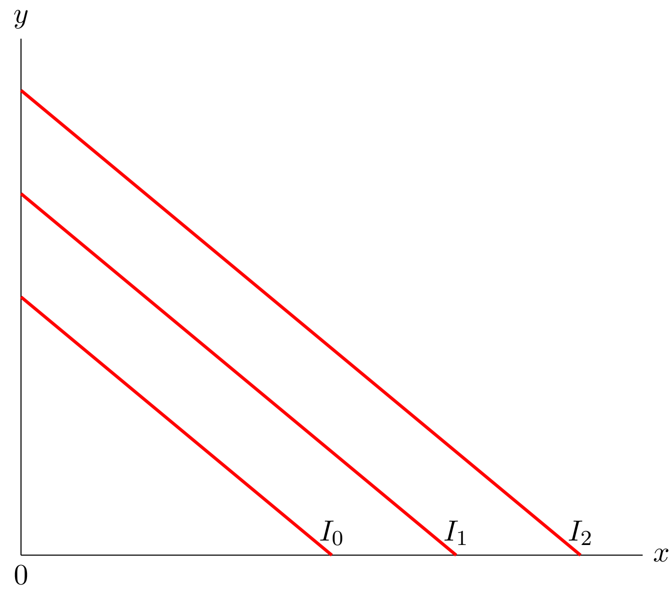
Finding the MRS of this function is as simple as taking the partial derivatives with respect to x and y. Perfect substitutes are often have the easiest MRS out of all the utility functions we study in this class. Going back to the example of dimes and nickels, the MRS would be

$$MU_x = 2$$

$$MU_y = 1$$

$$MRS = \frac{1}{2}$$

And voila. We now have a constant slope, as we should! I always am willing to trade two nickels for one dime, regardless of how many I have. A key takeaway here is that perfect substitutes have a constant MRS.



2.4.4 Perfect Complements

Perfect complements take on the form

$$u(x, y) = min\{\frac{1}{\alpha}x, \frac{1}{\beta}y\} \tag{6}$$

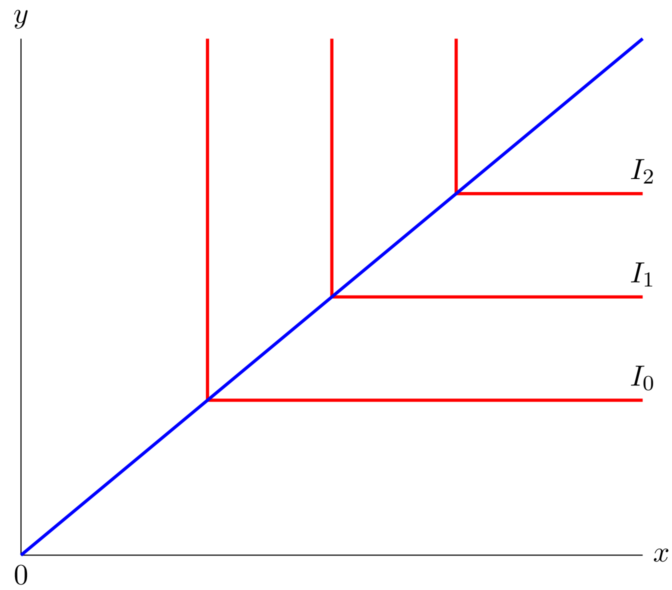
which produces L-shaped indifference curves. These consumers consume α unites of x for β units of y. The MRS of these functions is hard to find, because we cannot use calculus! Instead, we equate the inside terms, such that $\frac{1}{\alpha}x = \frac{1}{\beta}y$. This is what we call the kink point, or the optimal point of consumption. If you have trouble graphing these, I typically equate the inner terms and solve for y.

$$y = \frac{\beta}{\alpha}x$$

The result is a straight line through the origin on which your kink points will lie. Like the example before, say we have the utility function for a bicycle, $u = min\{\frac{1}{2}x, y\}$. Solving for y yields

$$y = \frac{1}{2}x$$

which is just a straight line containing our kinked points. These functions are not convex in the traditional sense, so we call them weakly convex.



y
I_2
I_1
I_0
0
x

Budget Constraints

Economics 100A

Fall 2021

1 Overview

1.1 Constraints

Economics 100A considers the "demand side" of the market. By this, we mean to say that we will be discussing the consumer and their choices. Eventually, we will be deriving demand curves using a combination of utility functions and budget constraints. So what exactly is a budget constraint?

A budget constraint represents how many goods a person can afford. When we model economic agents, we often think of these agents as maximizing their happiness (i.e., their utility) subject to some constraint. After all, even if having really expensive sushi for dinner every night would give me more utility than, for example, a sandwich, it is just not feasible for me to spend that much on sushi. This leaves me in a situation where I have to optimize my income to yield maximum happiness.

Now, in reality, there are potentially hundreds of lunch items from which I may choose. Rather than calculating optimal consumption between all of these goods, we will limit our analysis to just two. Suppose we have two goods, x_1 and x_2. Both of these goods have a respective price: p_1 and p_2. These prices are for each unit of the good, so two units of x_1 would cost $2p_1$. Importantly, we also have a certain amount of money to spend on these goods. In this class, we use I to represent income. So, we have x_1, x_2, p_1, p_2, and I.

If you recall from the math review notes, when we do constrained optimization, there are four considerations:

1. **Objective Function**: What are we maximizing/minimizing?
2. **Constraint**: What do we constrain our objective function to?
3. **Choice variables**: What variables do we choose to achieve maximization/minimization?
4. **Exogenous variables**: What variables affect the situation but are taken as given?

These notes are more concerned with the constraint, choice variables, and exogenous variables. As you might have guessed, our utility functions will be our objective functions. However, we will solve those later.

Constraint

In the constraint set up, we have our budget I and two goods: x_1 and x_2. These goods both have prices. For bundle (q_1, q_2) to be feasible, it must be the case that $p_1x_1 + p_2x_2 \leq I$. Now, this assumes that the consumer cannot borrow using credit, but we will cover those more complicated models in the future. For now, we have two goods and one period. To make our lives easier, we typically equate income and total expenditure.

Here is a question to test your understanding of expenditures and income. What if we had more than two goods? Say, we have N goods. What does our constraint look like?

$$p_1x_1 + ...p_Nx_N = \Sigma p_x x_i \leq I \tag{1}$$

Choice variables
What are the variables we can choose? Well, let's think about it. Can we choose the prices of the goods? Not really, considering we do not own the goods (yet!). It's tough to say that we can choose our income, though in theory we can allocate certain portions of our income to, say, lunch. The main thing we can control is how many of goods x_1 and x_2 we purchase.

Exogenous variables
Like previously stated, we cannot really choose our income or the prices. This makes some sense, as if we did have any say in these, we would make our incomes huge and reduce prices down to $0. Well, at least I would.

We want to remain simple in this class. Although we could allocate only part of our income to purchasing goods, we will assume in this class that a consumer allocates all of their income to goods x_1 and x_2, and that the consumer does not get to choose the prices. In other words, we take these as given in the models.

2 Drawing Budget Constraints

2.1 Overview

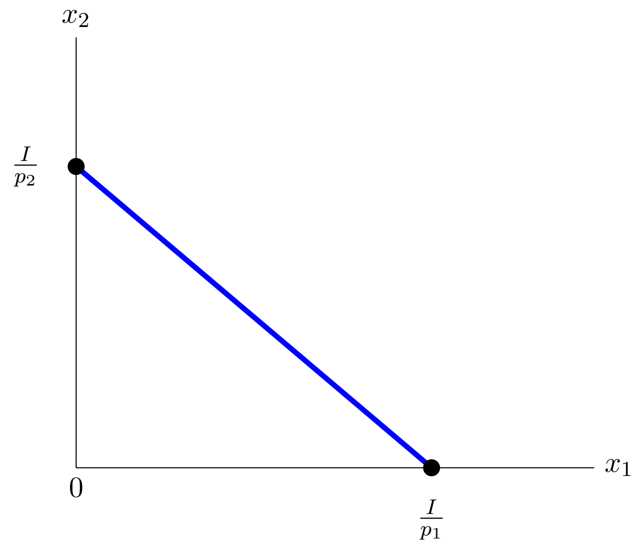
Recall that the equation for a budget constraint is

$$p_1x_1 + p_2x_2 = I \tag{2}$$

Now, how can we plot this equation in two dimensions? You should be comfortable with plotting linear functions, so I will assume you are familiar with the formula $y = mx + b$. We can actually rearrange the equation to write it in terms of x_2, since x_2 will be on the vertical (y) axis.

$$p_2x_2 = -p_1x_1I$$
$$x_2 = -\frac{p_1x_1I}{p_2}$$
$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{I}{p_2}$$

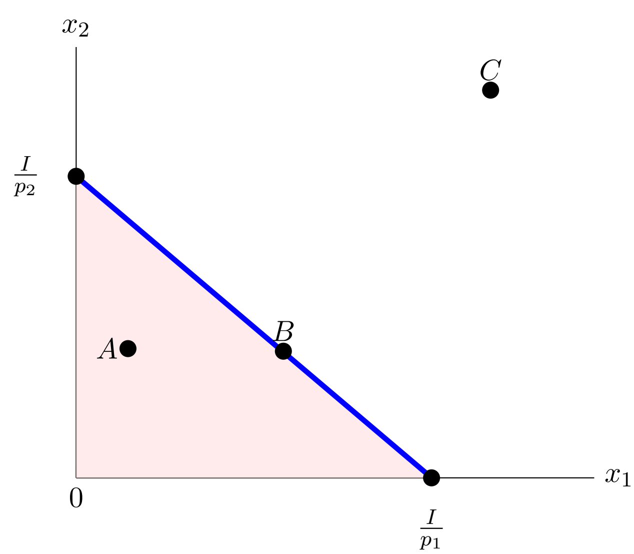
Now we see that if we want to graph a budget constraint, we first find the intercept at $\frac{I}{p_2}$ and draw the line according to the slope $-\frac{p_1}{p_2}$. For your own edification, you should prove that the x-intercept is just $\frac{I}{p_1}$.



In a relevant sense, the budget constraint limits the different bundles we can afford. In the diagram below, we have three points: A, B, and C. Notice how bundle A is below the constraint. This means that we can afford it, but it would not be optimal, as we are not spending all of our income. Anything shaded in red is affordable. However, in order to spend all of our income, we need to be consuming on the budget constraint. Thankfully, point B represents a consumer who spends all of their money. This consumer spends exactly all of their income. Finally, at point C, the consumer is spending more than their income, which in this class will pretty much be impossible.

To summarize:

1. **At point A**: The consumer is spending less than their income. $p_1x_1 + p_2x_2 < I$.
2. **At point B**: The consumer is spending exactly their income. $p_1x_1 + p_2x_2 = I$.
3. **At point A**: The consumer is spending more than their income. $p_1x_1 + p_2x_2 > I$.



Another more intuitive way of thinking about our values is that the intercept represents the maximum number of each good we can purchase. So, then, if we allocate all our our income I to a single good, it necessarily follows that we can only purchase $\frac{I}{p_x}$. If we are somewhere between the two extreme values, then we have some mix of goods that is purchased at either price. The slope here represents the change in y for a one-unit change in x. We tend to think about this as opportunity cost, as if we consume more of good x_1, we consume less of good x_2. You are giving up spending p_2 on x_2. Therefore, the slope is *necessarily* negative – since x_1 and x_2 are the only goods on which we can spend our money, it must be the case that increasing our purchases of x_1 means we spend less on x_2.

As you can probably guess, any point beyond (to the right of) the budget constraint is not feasible for the consumer. Any point below (to the left of) the budget constraint represents the consumer spending less than their income I. Any point on the line represents the consumer spending their entire income.

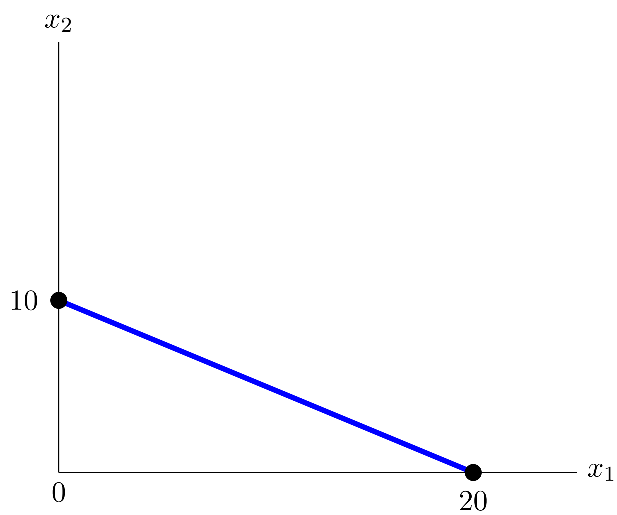
To summarize:

- **Any point to the right of the budget constraint** represents the consumer spending *more* than their income $p_1x_1 + p_2x_2 > I$
- **Any point to the left of the budget constraint** represents the consumer spending *less* than their income $p_1x_1 + p_2x_2 < I$
- **Any point to the on the budget constraint** represents the consumer spending *exactly all* their income $p_1x_1 + p_2x_2 = I$

2.2 Simple Case

In a sense, the budget constraint is the simplest graph in 100A because it is typically just a straight line. Each intercept will describe the maximum number of each good the consumer can purchase. This should make sense intuitively, as when $x_1 = 0$, x_2 is maximized and vice-versa. Let's do some simple examples.

Example one: A consumer chooses between apples (x_1) and oranges (x_2). p_1 is \$5 and p_2 is \$10. Income equals \$100. Draw the budget constraint and label the relevant points.



We can write out our constraint as follows:

$$100 = 5x_1 + 10x_2$$

Rearranging to solve for x_2:

$$x_2 = 10 - .5x_1$$

So now we see that our y-intercept will be at 10, and our constraint will have a negative slope of $-.5$. Alternatively, we could have just divided:

$$x_2 = \frac{I}{p_2} = \frac{100}{10} = 10$$

$$x_1 = \frac{I}{p_1} = \frac{100}{5} = 20$$

$$\frac{p_1}{p_2} = \frac{5}{10} = .5$$

Example two: A consumer chooses between coffee (x_1) and tea (x_2). p_1 is \$2 and p_2 is \$4. Income equals \$20. Draw the budget constraint and label the relevant points.

We can write out our constraint as follows:

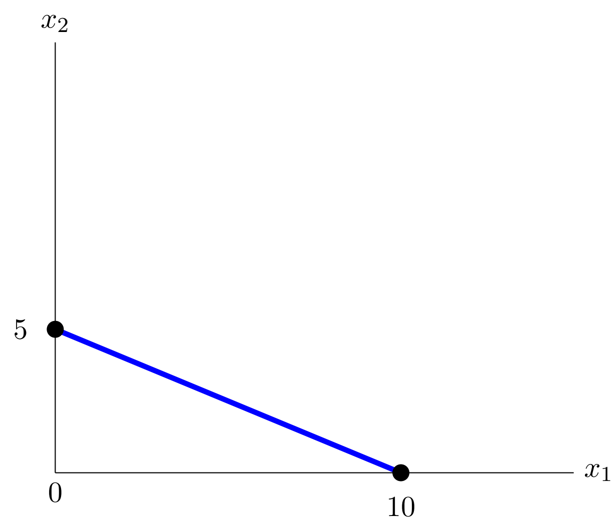
$$20 = 2x_1 + 4x_2$$

$$x_1 = \frac{20}{2} = 10$$

$$x_2 = \frac{20}{4} = 5$$

$$\frac{p_1}{p_2} = \frac{2}{4} = -2$$

Et voila. We have all we need to reasonably plot this thing!



Example three: A consumer can consume two bundles (q_1, q_2): $(8, 28)$ and $(12, 8)$. Draw the budget constraint and label the relevant points.

This is a fun, kind of cute one. We can find the slope between two points using the point-slope formula. Let's claim that $(8, 28) = (x_0, y_0)$. This means that $(12, 8) = (x_1, x_2)$.

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$m = \frac{8 - 28}{12 - 8}$$

$$m = \frac{-20}{4}$$

$$m = -5$$

So we have a slope of -5. Now what do we need? The intercept! Since we know that $y = mx + b$, we can plug in one of the points and find the y-intercept.

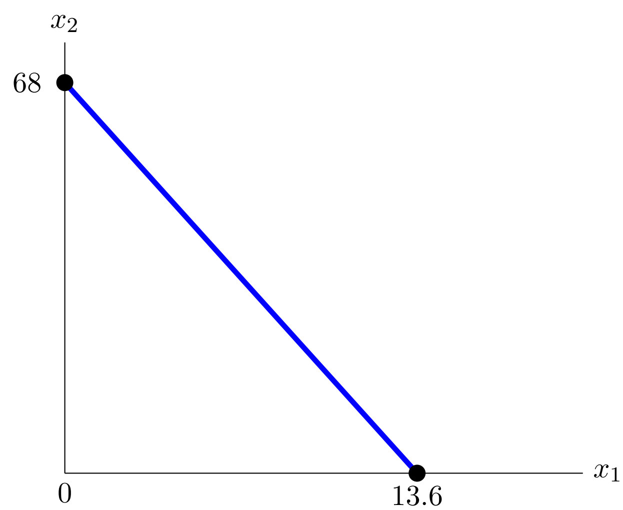
$$y = mx + b$$

$$8 = -5(12) + b$$

$$8 = -60 + b$$

$$b = 68$$

Wonderful! We found our y-intercept. We can now solve for the intercept, knowing that $y = -5x + 68$. We should get an x value of 13.6.



3 Change in Parameters

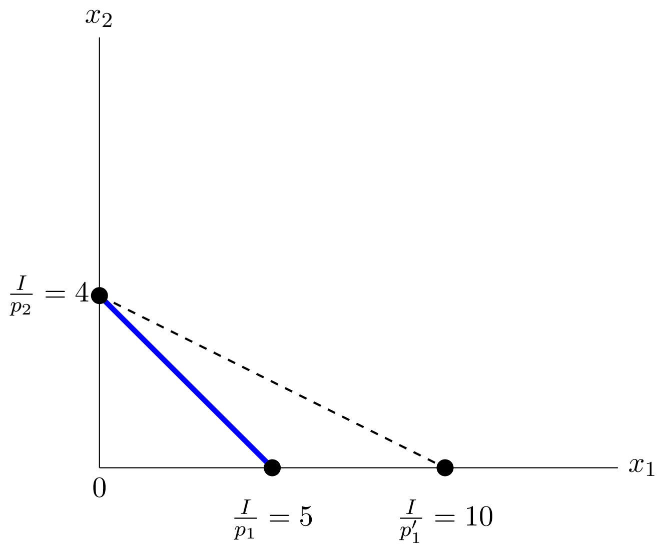
I think it is reasonable to imagine scenarios in which either prices or income change. How does the budget constraint shift, then? We can go over some specific examples first, but let's also think about how changes to our parameters change the math behind the constraint.

A change to income shifts the budget constraint out in a parallel fashion. This makes intuitive sense, because a consumer is either richer or poorer, but the slope of the line has not changed at all. In other words, if prices remain constant, the opportunity cost is the same for both goods.

A change in prices causes the budget constraint to pivot inward or outward. Convince yourself of this using a basic example. If the price of a good doubles, you can afford fewer of that good, meaning the intercept should shift inward/toward the origin. This represents a shift in the slope as well, since a slope parameter changes. Economically, think about this as the opportunity cost of consuming either good changing.

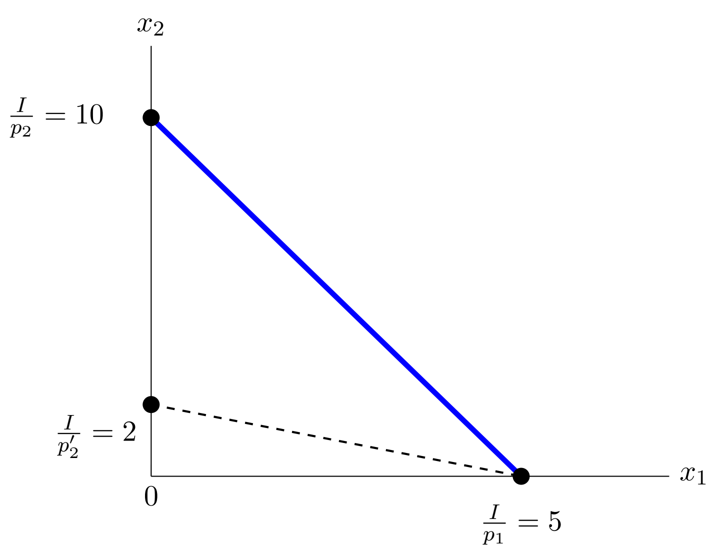
3.1 Examples

Example four: A consumer with an income of 20 chooses between two goods: x_1 and x_2. The corresponding prices are $p_1 = 4$ and $p_2 = 5$. Let's say that p_1 is reduced by 50% and falls to 2. Draw the new constraint.



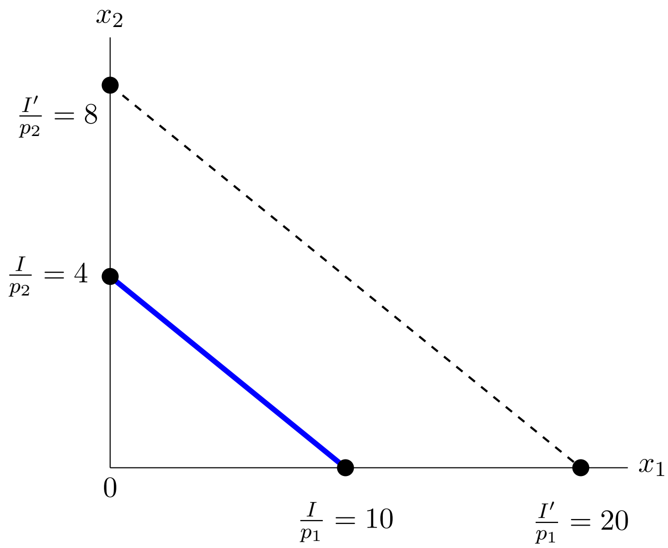
The slope in this case changes from $\frac{4}{20}$ to $\frac{2}{20}$, thereby making the constraint flatter. A flatter constraint means that we can afford more goods. Convince yourself of this. Do you see how this flatter, new constraint leads to more bundles being affordable? If not, email me.

Example five: A consumer with an income of 50 chooses between two goods: x_1 and x_2. The corresponding prices are $p_1 = 10$ and $p_2 = 5$. Let's say that p_2 is increased to 25. Draw the new constraint.



The slope in this case changes from $\frac{10}{5}$ to $\frac{10}{25}$, thereby making the constraint much, much steeper. A steeper constraint means that we can afford fewer goods. Convince yourself of this. We are restricting the number of bundles a consumer can afford.

Example six: A consumer with an income of 40 chooses between two goods: x_1 and x_2. The corresponding prices are $p_1 = 4$ and $p_2 = 10$. Let's say that our consumer wins the lottery and doubles their income.. Draw the new constraint.



Recall what we said earlier: if the income changes, we have a parallel shift. The slope is not changing at all. Therefore, we expect the budget constraint to pivot outward. Since our income doubled, we see that the number of goods we can afford in either direction (x and y) doubles as well.

4 Harder Parameter Changes

4.1 Non-linear Prices

The above examples are the most common in this class, so make sure to familiarize yourself with them. This section really concerns a few odd cases that are – in my opinion – much more realistic and interesting. The principles of the budget constraint are the same; we just need to get a bit more creative when thinking about how changes in price levels at different quantities affect the budget constraint.

Let's start by saying that $p_1 = \$3$ for the first 10 units of x. After the consumer purchases 10 units, the price increases to \$4. Let's slowly think through what patterns emerge when we change quantity purchase. If the consumer buys 5 units of x, they will pay $5x\$3 = \$15$. If they buy 10 units of x, they pay 10 x \$3 = \$30. If they buy 11 units of x, they pay $10x\$3 + 1x\$4 = \$34$. If they buy 12 units of x, they pay $10x\$3 + 2x\$4 = \$38$. Notice the pattern that develops? *Only after the 10 units* does the consumer face new prices, and accordingly, a new slope. This change in slope gives us a kinked budget constraint. Let's solve this one through: set $p_2 = 10$ and $I = 100$.

If we had constant prices, our budget constraint would be simple. It would have an intercept at $\frac{100}{3} = 33.33$ on the x-axis and an intercept at $\frac{100}{10} = 10$ on the y-axis. The slope of this line would be $\frac{-3}{10} = -0.3$.

What about after the price switch? Well, we already know that prices change *after* the tenth good, meaning we need to account for the fact that the consumer has already purchased 10 units of x at

$p_1 = 3$. This also implies that they have spent 10 x \$3 = \$30 already. Therefore, we write the new budget constraint like so:

$$4(x_1 - 10) + 3 * 10 + 10x_2 = 100$$
$$4x_1 - 40 + 30 + 10x_2 = 100$$
$$4x_1 + 10x_2 = 110$$

In general, the formula to account for changes in price (where new price is denoted by p') is:

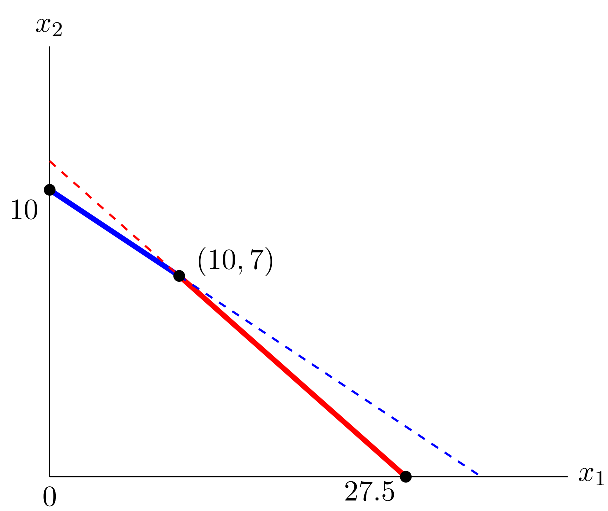
$$p'(x_1 - \bar{x}_1) + p_1 * x_1 + p_2 * x_2 = I \tag{3}$$

Where is the kink, though? We know that at some point, the slope must change. Well, we already know at which x_1 value the slope changes. To find the corresponding x_2 value, though, we need to plug in the value of x_1 at which the slope changes.

$$p_1x_1 + p_2x_2 = I$$
$$30 + 10x_2 = 100$$
$$10x_2 = 70$$
$$x_2 = 7$$

This means that we have a kink at the point $(10, 7)$. The initial slope is $-.3$, and the new slope is $\frac{4}{10} = -.4$. Likewise, from $4x_1 + 10x_2 = 110$, we know that we should have intercepts at $(\frac{110}{4}, \frac{110}{10})$.

The plot below shows the relevant budget constraint.



Optimal Choice

Economics 100A

Winter 2021

1 Overview

1.1 Prolegomenon to Any Future Optimization

Optimization is what you will be doing in this class and in most economics classes. The idea is simple: how do we maximize some good subject to some constraint? This is an idea that permeates all economics courses, since we are obsessed with efficiency, so now is the time to get used to optimization.

How will we be discussing optimization in this specific class, though? So far, we have set up how the consumer can rank bundles (utility function) and how the consumer is limited in what they can afford (budget constraints). We will put these two together to maximize our utility subject to some constraint. Remember, in each section I have been outlining the different components of optimization.

1. **Objective Function**: What are we maximizing/minimizing?
2. **Constraint**: What do we constrain our objective function to?
3. **Choice variables**: What variables do we choose to achieve maximization/minimization?
4. **Exogenous variables**: What variables affect the situation but are taken as given?

By now you can probably guess what our objective function and constraint are. But think a bit about what we want to maximize/minimize right now, and think about what information we are being given. Below, I will outline the specifics for our optimization.

1. **Objective Function**: We maximize consumer utility
2. **Constraint**: We constrain utility to some budget constraint.
3. **Choice variables**: We maximize utility via consuming x and y, so we maximize the quantity of each good.
4. **Exogenous variables**: We take prices and income as given.

Let's write the problem as follows:

$$maxu(x,y)$$

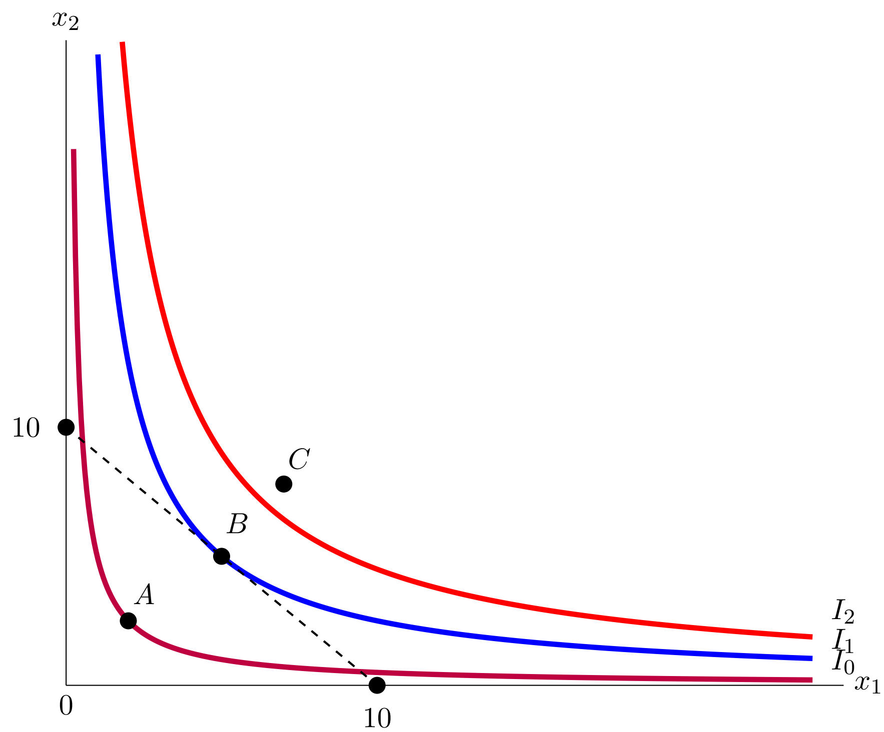
$$\text{s.t.} p_x x + p_y y = I$$

But what does this mean in words? We are trying to find the perfect indifference curve to optimize our utility. Since indifference curves exist everywhere, we want 1) the one which gives the consumer the most utility and 2) one that is feasible. If you had an unlimited budget, this problem would be completely meaningless. However, realistic budgets and prices make this a lot of fun.

In sum, we want to find the optimal bundle x^*, y^* which is feasible (affordable) and gives the consumer the most utility. If there is an indifference curve associated with a high utility, we assume that if the consumer does not take it, it must not be feasible.

2 Optimal Bundles

2.1 Graphical Solutions



The example above shows us why the blue indifference curve I_1 represents the optimal choice for a consumer. We have here that bundle B allows us to reach the highest feasible indifference curve. C is too expensive despite giving us higher utility. A on the other hand is cheaper but provides much less utility.

In summary, we can see that the optimal bundle occurs at a point of tangecy between the budget line and the indifference curve. This is where the slope of the budget constraint $\frac{p_x}{p_y}$ equals the slope of the indifference curve $\frac{MU_x}{MU_y}$. Why does this make sense? Remember that the MRS tells us how much y the consumer is willing to give up for one additional good of x. So, an MRS of 4 means that a consumer is willing to give up 4 units of y for one unit of x. But the slope of the budget constraint is just the opportunity cost of consuming one more of good x. Therefore, if $\frac{p_x}{p_y} < \frac{MU_x}{MU_y}$, the consumer could give up three units of y, get one more x and be better off than before. Since they only lose 3 units of y, they are better off than before, because they could have lost 4 units and been indifferent.

2.2 Lagrange and Substitution Methods

Lagrange To solve for the optimal bundle using constrained optimization, we can set up a Lagrangian:

$$\mathcal{L} = u(x, y) + \lambda(I - p_x x - p_y y)$$

For those who may need a calculus review, this equation allows us to maximize the values of x and y subject to a constraint. In this case, we are maximizing our utility from goods x and y and constraining ourselves to a budget. Solving for the optimal bundle first requires us to take three first order conditions.

In case you wanted to see something cool: let's look at the algebraic solution for finding the optimal bundle using the tangency condition and the Lagrange. Remember, we write out the Lagrange by

$$\frac{d_{\mathcal{L}}}{d_x} = \frac{\delta u(x,y)}{\delta x} - p_x \lambda = 0$$

$$\frac{d_{\mathcal{L}}}{d_y} = \frac{\delta u(x,y)}{\delta y} - p_y \lambda = 0$$

We can rearrange this and divide the first by the second:

$$\frac{d_{\mathcal{L}}}{d_x} = \frac{\delta u(x,y)}{\delta x} = p_x \lambda$$

$$\frac{d_{\mathcal{L}}}{d_y} = \frac{\delta u(x,y)}{\delta y} = p_y \lambda$$

$$\frac{\delta u(x,y)/\delta x}{\delta u(x,y)/\delta y} = \frac{p_x}{p_y}$$

And this is just the tangency condition! Let's rewrite this in terms of marginal utilities.

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

From here, we typically solve for the equation in terms of x or y, then we plug that into the budget constraint before solving for x or y.

3 Cobb-Douglas Utility Functions

3.1 General Notes and Results

1. Utility representation: $u(x, y) = x^{\alpha}y^{\beta}$
2. Indifference curves: standard convex curves that never touch the axes
3. Preferences: They prefer mixing goods and having a bit of both.

A Cobb-Douglas utility function typically takes the form

$$u(x, y) = x^{\alpha}y^{\beta}$$

Where x denotes good one and y denotes good two. It can also be written out as

$$u(x, y) = x^{\alpha}y^{\beta}$$

depending on the circumstances (e.g., if $\alpha + \beta > 0$, there is an interesting economic interpretation). I will go over both how to solve for the optimal bundle using a Lagrangian and the substitution method. Most students prefer the substitution method anyway, so it's important to be comfortable with it for exams. The Lagrangian is more useful for harder optimization problems, though, so if you intend on going further into the mathematical side of economics, you should try the Lagrange.

$$\begin{aligned}
\mathcal{L} &= x^{\alpha}y^{\beta} + \lambda(I - p_x - p_y) \\
\frac{d_{\mathcal{L}}}{d_x} &= \alpha x^{(\alpha-1)}y^{(\beta)} - p_x\lambda = 0 \\
\frac{d_{\mathcal{L}}}{d_y} &= (\beta)x^{(\alpha)}y^{(\beta-1)} - p_y\lambda = 0 \\
\frac{d_{\mathcal{L}}}{d_\lambda} &= I - p_x - p_y = 0
\end{aligned}$$

Note 1. We take the first order conditions of each unknown in order to maximize each variable. In your case, you will typically be given the income and prices, meaning you must solve for three unknowns: x, y, and λ.

Once we have our first order conditions, I like to solve for X and Y in terms of *lambda*. We do this by swinging our negative lambda over to the other side of the equation and simplifying until we can equate the two equations.

$$\begin{aligned}
\alpha x^{(\alpha-1)}y^{(\beta)} &= p_x\lambda \\
\beta x^{(\alpha)}y^{(\beta-1)} &= p_y\lambda
\end{aligned}$$

From here, we can simplify x in terms of y or the other way around. Do whatever is easiest for you. In any case, we have to divide by the prices of each to get the following:

$$\frac{\beta x^{(\alpha)} y^{(\beta-1)}}{p_y} = \frac{\alpha x^{(\alpha-1)} y^{(\beta)}}{p_x}$$

Once we have our equality, we can just solve for x or y. In this case, I will solve for y, but I encourage you to solve for X to get better mathematical practice.

$$y = \frac{\beta}{\alpha} \frac{(p_x)x}{p_y}$$

Once we have y, we can find our tangent point on the budget constraint by substituting $\frac{\beta}{\alpha} \frac{(p_x)x}{p_y}$ for y. We would get something like this:

$$I = p_y(\frac{\beta}{\alpha} \frac{(p_x)x}{p_y}) + p_x(x)$$

This thing looks disgusting, I know, but we can take it one step at a time. If you are like me and hate algebra, just slowly simplify until we can write this out as a nice relationship. Luckily for us, the p_y can cancel in the denominator, making things a bit easier.

$$I = \frac{\beta}{\alpha} p_x(x) + p_x(x)$$

Factor out prices and good x.

$$I = (1 + \frac{\beta}{\alpha}) p_x x$$

$$I = (\frac{\alpha + \beta}{\alpha}) p_x x$$

$$I = \frac{p_x x}{\alpha}$$

$$x = \frac{\alpha}{\alpha + \beta} \frac{I}{p_x}$$

We see now that the consumer's consumption of good x relies on three things: income, exponents (α and β), and the price of good x. Notice, then, that a change in the price of good y will not change the consumer's consumption of good x. It only changes the consumption of good y.

Since we already have a ratio of x to y, we can just solve for y by plugging in the variables. Remember, we said that:

$$y = \frac{\beta}{\alpha + \beta} \frac{(p_x)x}{p_y}$$

So we can find y once we have x, which is $\frac{I\alpha}{p_x}$. I will leave finding y as an exercise for you.

Substitution The tangency condition tells us that

$$MRS = MRT$$
$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

As we saw from the graph above, the tangency requires the slopes to be equal. In this case, we set the slope of the indifference curve equal to the slope of the budget constraint. Let's do a quick example for a simple utility function subject to a general budget constraint:

$$u(x, y) = xy$$

$$I = p_x x + p_y y$$

Hopefully by now you are comfortable with finding the MRS. We write the MRS as the ratio of marginal utilities:

$$MRS = \frac{MU_x}{MU_y} = \frac{y}{x}$$

After we have the MRS, we equate it to the slope of the budget constraint:

$$\frac{y}{x} = \frac{p_x}{p_y}$$

Now what we want to do is solve a system of two equations with two unknowns, x and y (assume that prices are given to us and therefore we cannot solve them). We must now solve for this equation in terms of either x or y.

$$y = \frac{p_x x}{p_y}$$

After this, we have to plug this into the constraint to maximize the value of either x or y. Let's do that now:

$$I = p_y(\frac{p_x x}{p_y}) + p_x x$$

See that we can cancel p_y out and factor like terms:

$$I = p_x x + p_x x$$
$$I = x(p_x + p_x)$$
$$I = x(2p_x)$$

And now we divide to solve for x.

$$x = \frac{I}{2p_x}$$

And now since income and prices are exogenous, we just plug those in to solve for the optimal quantity. But what about y? We know from previous algebra that

$$y = \frac{p_x x}{p_y}$$

so we can plug in $x = \frac{I}{2p_x}$ so solve for y. I will leave this as an exercise for you. You should get that

$$y = \frac{I}{2p_x}$$

3.2 Example

Suppose we have

1. $u(x, y) = x^3 y^2$
2. $I = 80, p_x = 2, p_y = 4$

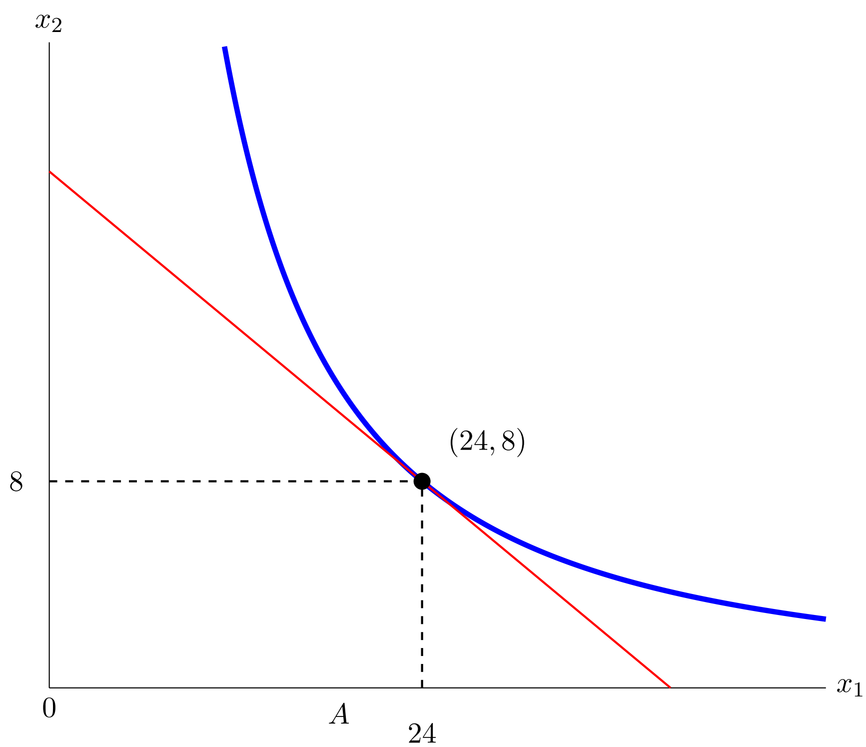
You can do a Lagrangian but it would take a long time. Instead, set up MRS = MRT. Let's start by

$$MRS = \frac{3}{2}\frac{y}{x}$$
$$MRT = \frac{2}{4} = \frac{1}{2}$$
$$\frac{3}{2}\frac{y}{x} = \frac{1}{2}$$
$$\frac{2x}{3y} = 2$$
$$x = 3y$$

Plug this into the budget constraint:

$$p_x x + p_y y = I$$
$$2(3y) + 4y = 80$$
$$2(3y) + 4y = 80$$
$$10y = 80$$
$$y = 8$$

C

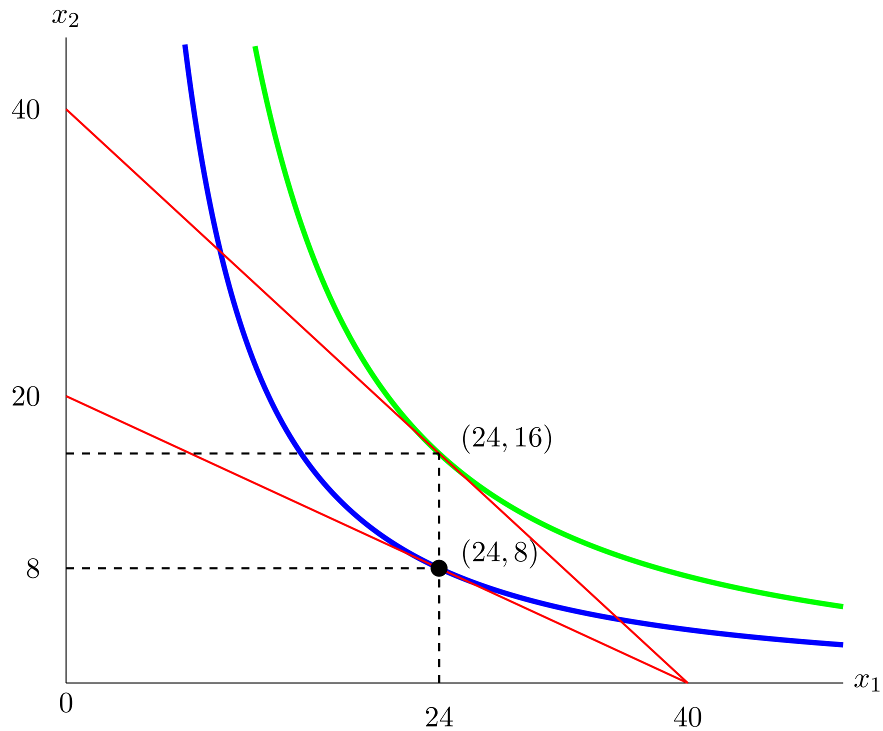


And through the $x = 3y$ relationship, we know that $x = 3(8) = 24$.

But let's take this a step further: what if p_y fell to 2? How would this change the optimal bundle? We know from above that $y = \frac{\beta I}{\alpha + \beta p_y}$, so I will skip the optimization steps for now and write the results:

$$y = \frac{\beta}{\alpha + \beta} \frac{I}{p_y} = \frac{2}{5} \frac{80}{2}$$
$$y = 16$$

And there we have it! When the price of y halves, the quantity demanded doubles. And since neither I nor p_x changed, x is still 24. A graph of this change would look something like this:



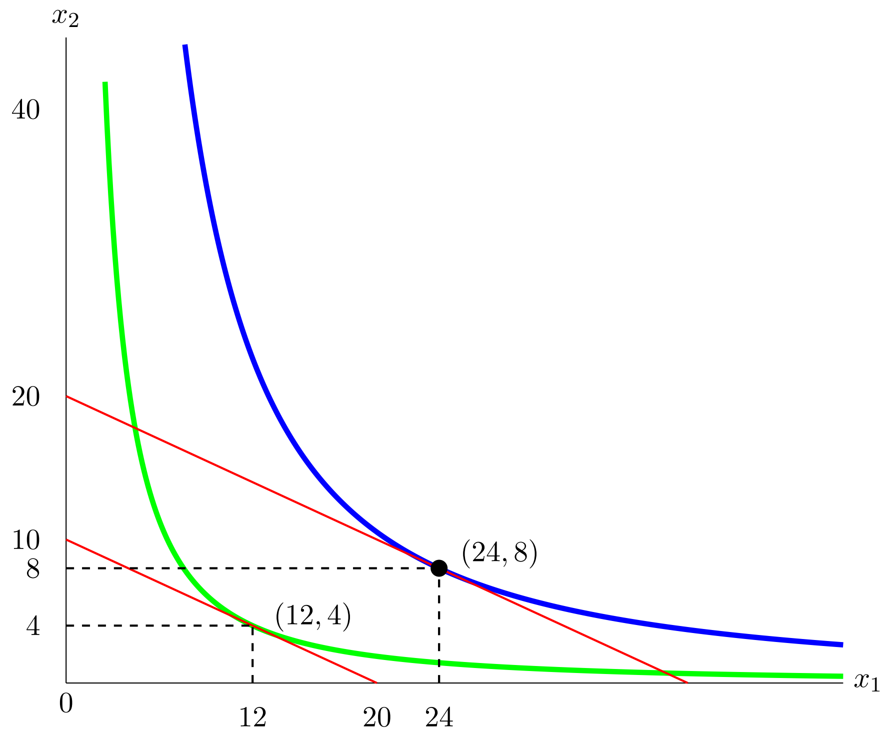
So this just shows that our consumption of good y changes whereas our consumption of good x remains constant. But what if instead of a change in prices, we faced a change in income? Let's say that income decreases from 80 to 40. What happens to x^* and y^*?

Using the shortcut for both, we see that

$$x^* = \frac{\alpha}{\alpha + \beta}\frac{I}{p_x} = \frac{3}{5}x\frac{40}{2} = 12$$

$$y^* = \frac{\beta}{\alpha + \beta}\frac{I}{p_y} = \frac{2}{5}x\frac{40}{4} = 4$$

When our income halves, our consumption of each good also halves. Let's look at this graphically:



This graph shows the contraction of income and the subsequent decrease in consumption of both goods. Consumption of x and y changed proportionally with income.

4 Perfect Substitutes

4.1 General Notes and Results

1. General form: $u(x, y) = \frac{1}{\alpha}x + \frac{1}{\beta}y or u(x, y) = \alpha x + \beta y$
2. Indifference Curves: Linear indifference curves
3. Preferences: A consumer is as happy with α units of good x as they are with β units of y.

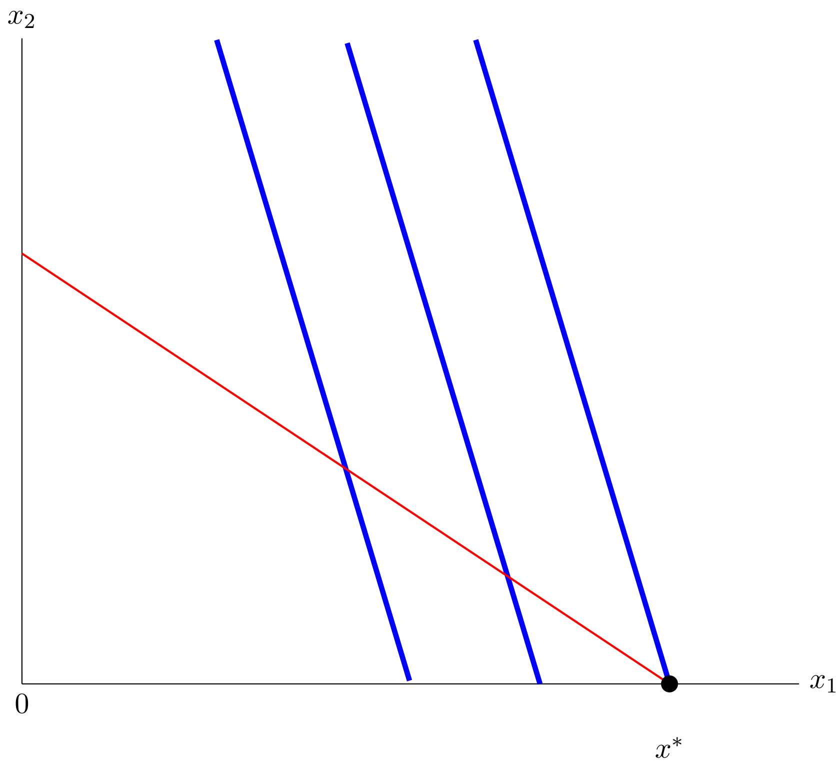
Perfect substitutes are utility functions whose MRS are some constant. Typically, you can identify a perfect substitutes function with ease; however, there are some exceptions. (Try taking the MRS of $\sqrt{2x - y}$ for example.) Using constrained optimization for perfect substitutes is a complete waste of time, so don't even bother. Remember, since the MRS is constant, we are never going to find a point of tangency. Instead, we are going to find a corner solution.

Since both marginal utilities are positive and constant, and prices are constant, we should just find which marginal utility constantly yields higher utiliy per dollar spent. In short, we compare

$$\frac{MU_x}{p_x} vs \frac{MU_y}{p_y}$$

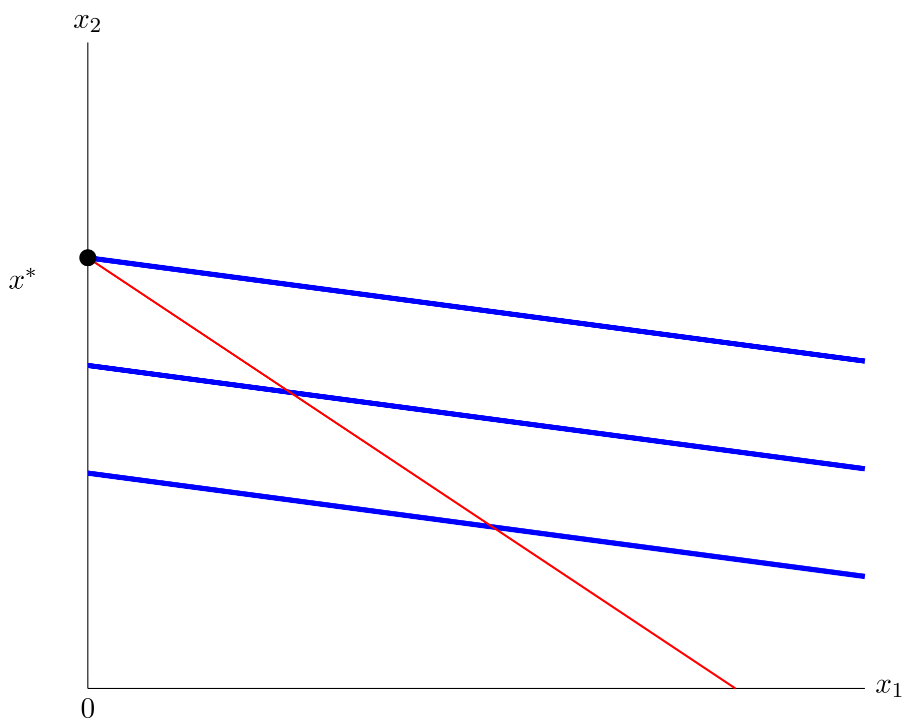
We must compare the marginal utility per dollar in order to find the optimal bundle. The best bundle will be the one that provides higher utility per dollar.

Let's go through some scenarios.If $\frac{MU_x}{p_x} > \frac{MUy}{p_y}$, we will have a graph that looks something like this:



This means that we have a solution on the x-axis, where $x^* = \frac{I}{p_x}$ and $y^* = 0$. Look how additionally the indifference curves are steeper than the budget constraint. This should give you another deeper intuition behind why we have a corner solution: we can reach a higher indifference curve by consuming only x.

If $\frac{MU_x}{p_x} < \frac{MUy}{p_y}$, we will have a graph that looks something like this:



This means that we have a solution on the y-axis, where $y^* = \frac{I}{p_y}$ and $x^* = 0$. Look how additionally the indifference curves are less steep than the budget constraint. We can reach a higher indifference curve by consuming only y.

The only possibility is that $\frac{MU_x}{p_x} = \frac{MUy}{p_y}$, which would mean that the slopes of the budget constraint and the indifference curves are the same. Additionally, the marginal utility per dollar spent is the same for both goods. Any point on the indifference curve is optimal.

4.2 Examples

Say we have $u(x, y) = 10x + 5y$ and $p_x = 5, p_y = 2$. Additionally, $I = 20$. Comparing the marginal utility per dollar is simple at this point, so let's just do it:

$$\frac{10}{5} < \frac{5}{2}$$

So the consumer consumes $y = \frac{20}{2} = 10$ units of y and zero x.

If the price of y increases from 2 to 4, though, we have a different story. Comparing marginal utility per dollar:

$$\frac{10}{5} > \frac{5}{4}$$

so the consumer buys $\frac{20}{5} = 4$ and 0 units of y.

5 Perfect Compliments

5.1 General Notes and Results

1. General form: $u(x,y) = min\frac{1}{\alpha}a\frac{1}{\beta}y$
2. Indifference Curves: L-shaped indifference curves
3. Preferences: A consumer who must consume bundles in specific proportions. Specifically, they consume α units of x with β units of y.

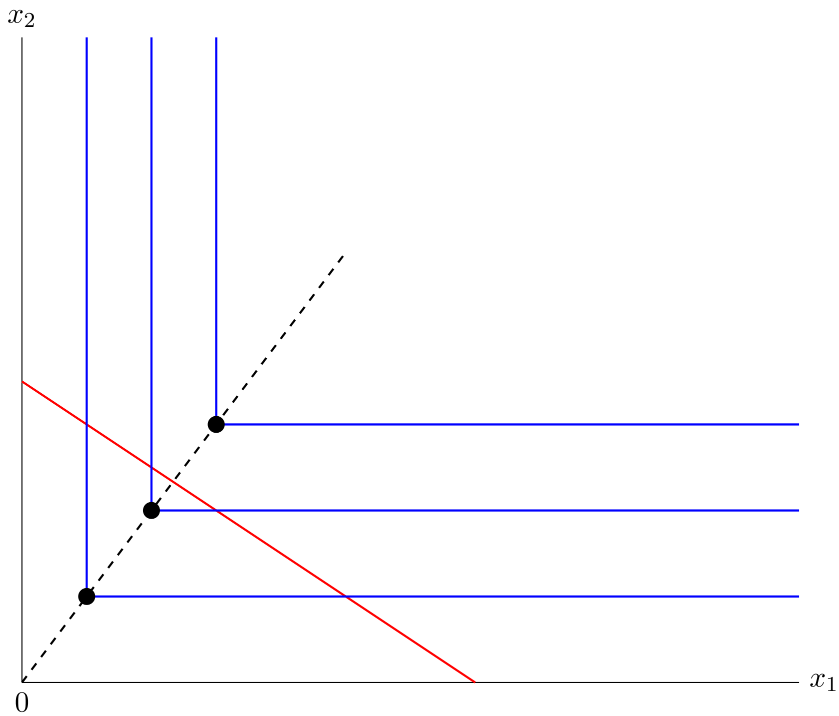
Now, we do not use the tangency condition with perfect complements. Think about it: what is the slope (MRS) of the indifference curve? The vertical line has a slope of $-\infty$, whereas the flat line has a slope of 0. And what is the slope of the kink? It's very strange to think about. This function is not differentiable, so we have to approach it with a different strategy.

This is why we have to think about optimal consumption. Think about this kink in terms of *optimal* bundles. If we consume at each corner, we are not wasting any goods, so to speak. If we deviate from the kink, we have an MRS of either ∞ or 0. We are essentially buying goods that give us no marginal utility. Now, let's think about where the kinks are. They occur when we equate the two inner components:

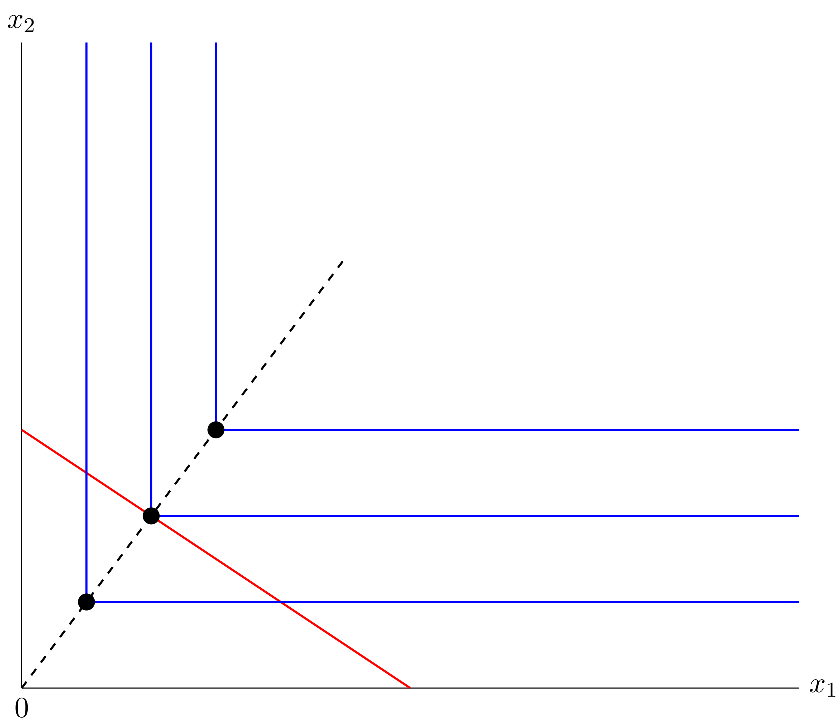
$$\frac{1}{\alpha}x = \frac{1}{\beta}y$$
$$\beta x = \alpha y$$
$$y = \frac{\beta}{\alpha}x$$

How wonderful is it that we can plot this line? On this line, all the optimal kink points will occur. Okay, so let's just solve for the optimal bundles by plugging that value of y into the budget constraint.

$$p_1 x + p_y y = I$$
$$p_1 x + p_y(\frac{\beta}{\alpha}x) = I$$
$$x(\frac{\alpha p_1 + \beta p_2}{\alpha}) = I$$
$$x = (\frac{\alpha I}{\alpha p_1 + \beta p_2})$$



Notice in the above case, the budget line is not tangent with any of the kink points. Additionally, the slope of the dashed line is just $y = \frac{\beta}{\alpha}x$. Let's look at a case where the line is tangent.



I should note that technically there is not a "tangency" in the strict sense. The slope is undefined at the kink point. However, it is the optimal point.

5.2 Example

Let's assume that we have $u = min(x, 5y)$ and $p_1 = 3, p_2 = 1, I = 15$. How do we find the optimal bundle? Let's first equate the inner terms:

$$x = 5y$$
$$y = \frac{1}{5}x$$

Plugging x into the budget constraint:

$$x = 5y$$
$$p_x x + p_y y = I$$
$$p_x(5y) + p_y y = I$$
$$y(5p_x + p_y) = I$$
$$y = \frac{I}{(5p_x + p_y)}$$

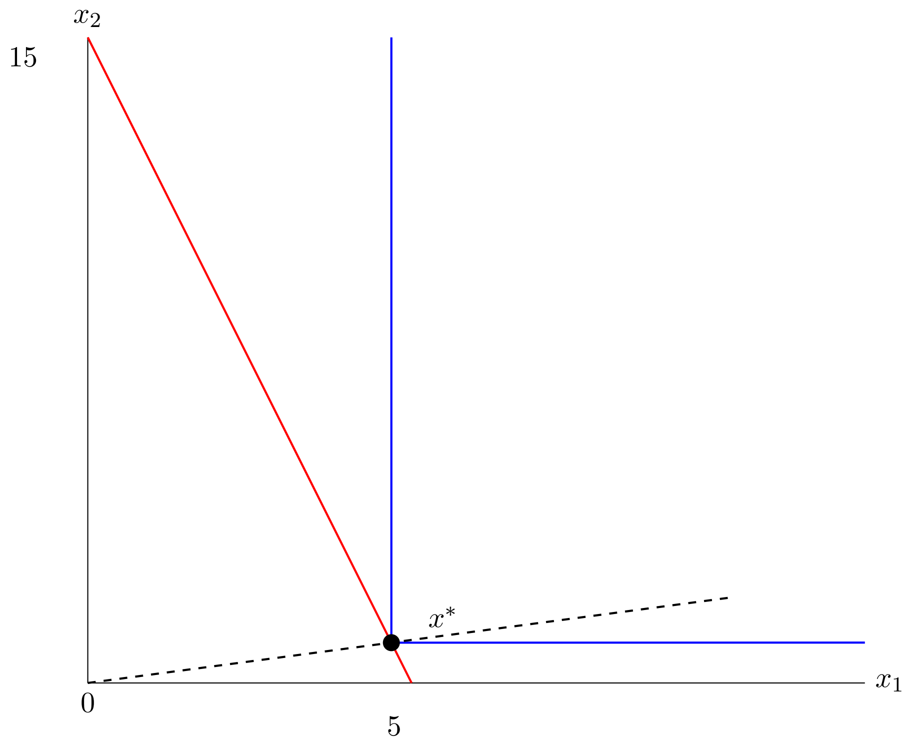
And from the previous $x = 5y$, we know that

$$x = \frac{5I}{(5p_x + p_y)}$$

Now all we have to do is plug in the parameters p_x, p_y, and I.

$$x = \frac{5I}{(5p_x + p_y)} = \frac{5(15)}{5(3) + 1} = \frac{75}{16}$$
$$y = \frac{I}{(5p_x + p_y)} = \frac{15}{5(3) + 1} = \frac{15}{16}$$

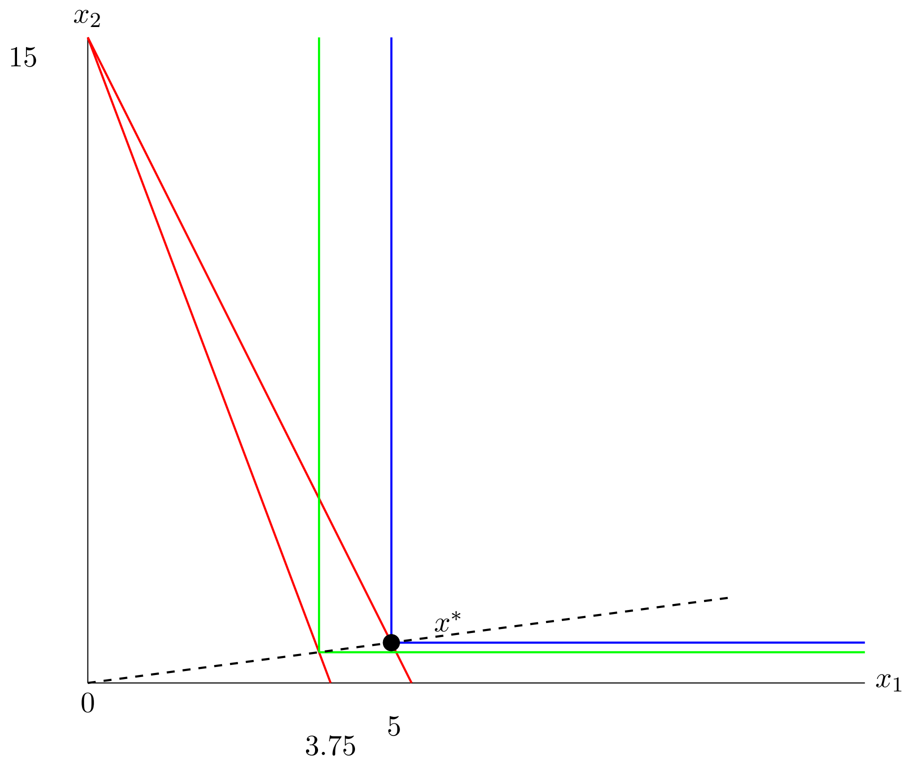
Sometimes your numbers won't be clean, but that's okay! In any event, we can visual these results:



Quickly, let's assume that p_1 increases to 4. Let's solve for the optimal bundles.

$$x = \frac{5I}{(5p_x + p_y)} = \frac{5(15)}{5(4)+1} = \frac{75}{21}$$

$$y = \frac{I}{(5p_x + p_y)} = \frac{15}{5(4)+1} = \frac{15}{21}$$



The above graph is not the best, I recognize, but the intuition is just to see how changes in the budget constraint changes the optimal bundle.

6 Quasi-Linear

6.1 General Notes and Results

1. General Form: $u(x1, y) = f(x) + y$, where $f(x)$ is some nonlinear function of x and y is some linear term.
2. Indifference Curves: Wider curves that tend to be either steeper or flatter than Cobb-Douglas curves. They can touch the axes. A key property is that each indifference curve is just a parallel shift of the other indifference curves.
3. Preferences: Mixer whose MRS is dependent only on x, and not at all on y.

We write quasi-linear utility functions in the following form:

$$u = f(x) + y$$

These are called quasi-linear because of the linear term for good y. We will explore the components of this function in due time. However, let's first return to the fun that is constrained optimization.

$$\mathcal{L} = f(x) + y + \lambda(I - p_x x - p_y y)$$

First order conditions:

$$\mathcal{L} = f(x) + y + \lambda(I - p_x x - p_y y)$$

$$\frac{d_{\mathcal{L}}}{d_x} = f'(x) - \lambda p_x = 0$$

$$\frac{d_{\mathcal{L}}}{d_y} = 1 - p_y \lambda = 0$$

$$\frac{d_{\mathcal{L}}}{d_\lambda} = I - p_x - p_y = 0$$

Anyone else see just how beautiful this result is? What is the difference between the first order conditions here versus the Cobb-Douglas function? Here, $\lambda = 1/p_y$. Why is this remarkable? What does λ normally represent? It represents the marginal effect of income on utility. In this case, any change in utility is constant. This means that, as income increases, the agent simply purchases more of good y.

From here, we need to solve for x. That's typically a pain to do in the abstract, general case. Plus, it may help to go through an actual example. So let's say that $u = x^{\frac{1}{2}} + y$. How would we solve this? I will skip a few steps, but I will provide key steps.

$$\lambda = \frac{\frac{1}{2}x^{\frac{-1}{2}}}{p_x}$$

$$\lambda = \frac{1}{p_y}$$

$$\frac{1}{p_y} = \frac{\frac{1}{2}x^{\frac{-1}{2}}}{p_x}$$

$$\frac{p_x}{p_y} = \frac{1}{2}x^{\frac{-1}{2}}$$

$$x = (\frac{2p_x}{p_y})^{-2}$$

Wow! x is only reliant on the prices! This means that the consumer buys a "fixed" number of x (based on the price ratio) and spends the rest of their income on good Y. Any increase in income goes directly to good y. However, a change in prices will lead to an increase in both goods. This is a really important point, so hammer it home.

I suggest solving for y yourself for practice. However, you should get the result:

$$y = \frac{I}{p_y} - \frac{1p_y}{4p_x}$$

A rise in p_y is associated with less y. Is this surprising? Let's solve for I. We can do some clever algebra and get the following:

$$I = \frac{1p_y^2}{4p_x}$$

If income is greater than $\frac{1p_y^2}{4p_x}$, the consumer will buy both goods. However, if income is less than that, the consumer will only purchase good one and we will be left with a corner solution.

Let's rewrite this in terms of MRS. Suppose we have $u(x, y) = f(x) + y$. To find the tang ency condition, we equate MRS and the price ratio:

$$MRS = \frac{MU_x}{MU_y} = \frac{f'(x)}{1} = \frac{p_x}{p_y}$$

Notice how we have an MRS that only depends on x and not y. Interesting. Let's solve for x.

$$f'(x) = \frac{p_x}{p_y}$$

$$x = f'^{-1}\frac{p_x}{p_y}$$

At this point it's getting late in the notes, and I am getting tired. Hence the abuse of the notation. By f'^{-1} I mean the inverse derivative. Let's plug this result into budget constraint to solve for y.

$$p_x x + p_y y = I$$

$$p_y y = I - p_x x$$

$$y = \frac{1}{p_y}(I - f'^{-1}\frac{p_x}{p_y})$$

This basically says to us: find your optimal x from the optimality condition, and whatever you don't spend on x, put it all towards y. Since x does not depend on income, as your income increases, you spend a higher proportion of your income on y.

Demand

Economics 100A
Winter 2021

1 Overview

Let's once again restate the utility maximization problem:

$$maxu(x,y)$$

$$s.t.p_x x + p_y y = I$$

As you have seen in class, when we solve this problem for general parameters, we find the demand function. Well, as it turns out, this is called the **Marshallian demand**. We tend to write this as a function of prices and income:

$$x^*(p_x, p_y, I)$$

The choice variable here is quantity demanded and the parameters of the demand function are simply prices and income. Turns out if we change individual parameters, we can see some cool properties of demand functions. These notes do just that.

From the outset, I will try to make my notation as clear as possible. I will write functions in terms of their given parameters and the parameter we change. Also, because we fancy ourselves artists of some variety, we will have a lot of graphical interpretations. This means that we have to be careful about the difference between dependent and independent variables. Prices always go on the vertical (y) axis whereas quantities always go on the horizontal axis (except in the case when we graph quantities of x and y together, then those look like our standard graphs).

Let's summarize for sake of clarity:

1. Solve for the consumer's demand as a function of prices and income. In other words, $x(p_x, p_y, I)$.
2. Change *one* parameter of interest while holding all else constant
3. Re-solve the consumer utility maximization problem
4. Observe how x_i changes as the other parameters change. This should give you a new point on the demand curve
5. If you do this for many values of p_x, p_y, and I, you will get an **offer curve!**

2 Own-Price

It is always useful to have some end goal in mind when doing economics. This helps shape the method of our analysis, as well as guide our interpretation of our results. In the case of own-price demand, we ask: "What is the relationship between the price of a good and its demand?" This helps tremendously in a few ways. Firstly, it allows us to isolate what variable we want to see change: own price. When we look at own-price demand, we are seeing how changes in a good's price changes the demand for this good.

So, mathematically, what exactly are we doing? We are plotting $x(p_x, p_y, I)$ where I and p_y are held fixed. Remember, since this is economics, we must make ridiculous assumptions about the world, including that other prices and income do not change. In any case, when we plot x, we are getting a *demand curve*. How amazing is that?! You have been studying and drawing these things for ages, and now you get to see the foundations of them. The inverse of this curve also has an interesting name. Can you guess it? If you guessed *inverse demand*, then you are correct! Economists have no imagination except for their silly assumptions, it seems.

Let's do a simple example using Cobb-Douglas preferences.

2.1 Cobb-Douglas Demand

I am making a number of short-cuts because at this point I hope you are comfortable with constrained optimization. Recall that the demand for a Cobb-Douglas function can be written as follows:

$$x^*(p_x, I) = x(p_x, p_y, I) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{p_x}$$

We know that demand for good x does not rely on the price of y. What else do we see? Since the price of x is in the denominator, we can also see that as the price of x increases, the demand for good x decreases. This is just the law of demand! How splendid. But we are interested in plotting this relationship. And since prices are always graphed on the vertical axis, we have to solve for this function in terms of p (see the math review if you are not comfortable with this).

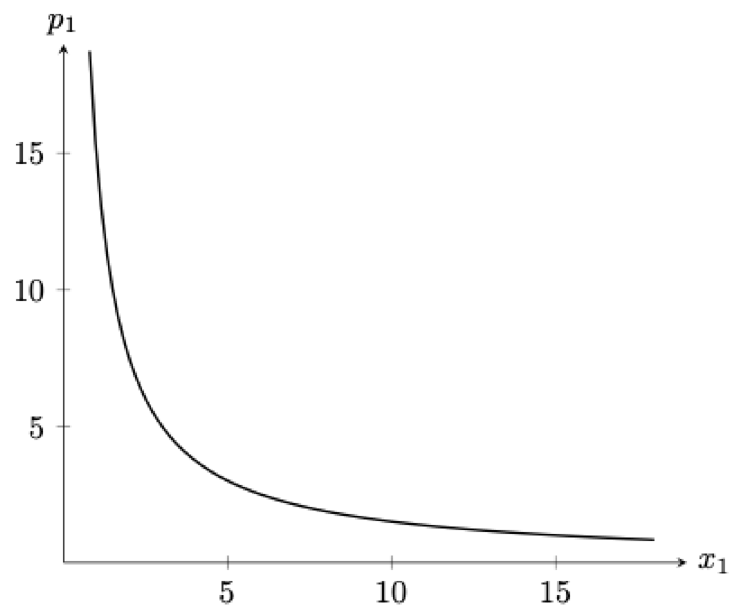
Our function should look like this:

$$p_x(x, I) = p_x(x, p_y, I) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{x}$$

Not a whole lot changes. I will leave the algebra to you as an exercise. If you want to see it, come to office hours. So, now that we have that, we can finally plot this thing. Let me give you some nice parameters so we can do that. Suppose that $\alpha = .75$, $\beta = .25$, $p_x = 3$, $p_y = 1$, and $I = 20$. If we plug these into our own-price demand for x, we get:

$$x^*(p_x, 1, 20) = \frac{3}{4} \cdot \frac{15}{p_x}$$

$$p_x(x^*, 1, 20) = \frac{15}{x^*}$$



These are the demand curves for x^*, but we could just as easily solve this for y^*. Using the shortcut yields:

$$y^*(p_y, 3, 20) = \frac{1}{4} \cdot \frac{20}{p_y}$$

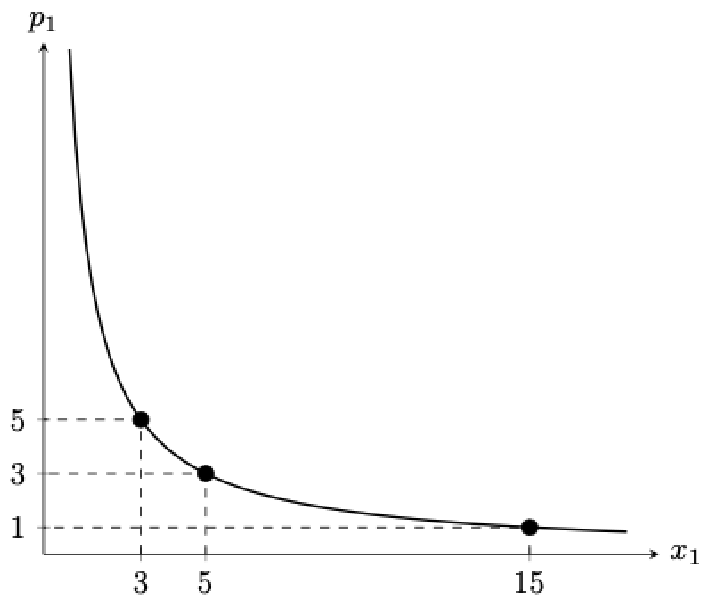
$$p_y(y^*, 3, 20) = \frac{5}{p_y}$$

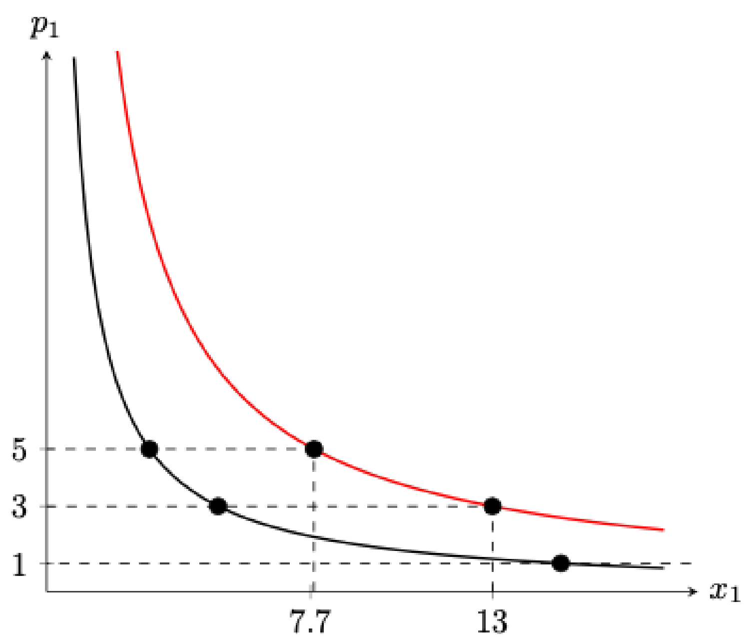
Let's take a moment to see if these demand functions make sense in the first place. Notice how we only have this relationship between prices and optimal quantity of either good. Just by looking at our functions, we see that as the price of x or y increases, we will have a smaller value for x^* and y^*. This is really just the law of demand! As prices increase, the consumer demands fewer of that good.

Let's try graphing this thing! If we plot the demand curve for x, we see a standard, downward-sloping demand curve. However, we must always remember to include p_x on the vertical axis. So, technically, this is an *inverse demand* curve.

Now, what happens when the price of x changes? What happens to demand if p_x=1? What if p_x=3? Or even $p_x = 5$? All we have to do is move along the demand curve to see the different quantities demanded at each different price! As we can see in the diagram, at a $p_x = 5$, the consumer consumes 3 x. Likewise, at a $p_x = 1$, the consumer consumes 15 x. We can verify this algebraically using our demand function above.

Next, suppose we would like to consider a change in income. Suppose that I increases to 52. What happens to demand when income changes? Well, as you should recall, there is an overall shift in the demand curve. As we see, the quantity demanded increases at each and every price level! This





gives us an outward shift in the demand curve. We can also see this using our demand function:

$$x^*(p_x, 1, 52) = \frac{3}{4} \cdot \frac{52}{p_x}$$

$$x^*(p_x, 1, 52) = \frac{39}{p_x}$$

Above, I graph the new curve in red.

As previously stated, the at each price level there is a corresponding higher level of demand.

Let's now look at the price offer curve.[1] Recall that for the offer curve, we must satisfy the following condition:

$$(x(p_x,p_y,I),y(p_x,p_y,I)) = (\frac{\alpha}{\alpha+\beta}\cdot\frac{I}{p_x},\frac{\alpha}{\alpha+\beta}\cdot\frac{I}{p_y})$$

Since we already have the exponents (alpha and beta), prices and income, we can just rewrite the expression above as:

$$(x(p_x,p_y,I),y(p_x,p_y,I)) = (\frac{15}{p_x},5)$$

Since these preferences are Cobb-Douglas, we should not be surprised to see that we have a constant value for y when we consider changes to the price of x. Now, let's be professional and graph this the "proper" way. Our Cobb-Douglas tangency condition was:

$$y = \frac{\beta}{\alpha}\cdot\frac{p_x}{p_y}x$$

Notice that there is a p_x in here. We want to get rid of this, so we are going to plug p_x into $p_x(x,p_y,I) = \frac{\alpha}{\alpha+\beta}\cdot\frac{I}{x}$. This yields:

$$y = \frac{\beta}{\alpha}\cdot\frac{1}{p_y}\cdot(\frac{\alpha}{\alpha+\beta}\cdot\frac{I}{x})\cdot x$$

$$\therefore y(x,p_y,I) = \frac{\beta}{\alpha+\beta}\cdot\frac{I}{p_2}$$

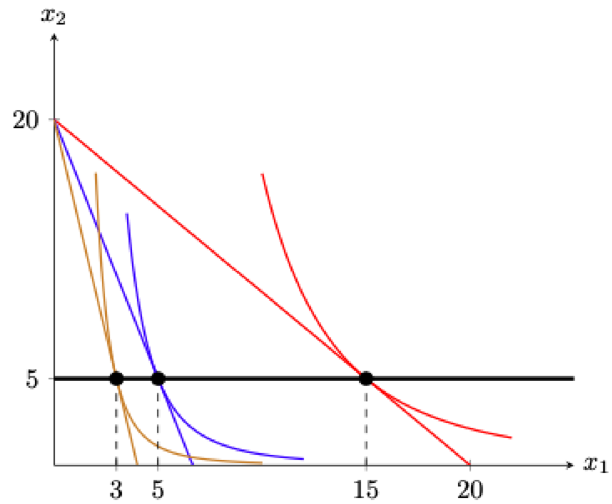
Now, we no longer have p_1 in the equation. This is what we wanted! We also do not have x. This is also expected, since we know that the optimal y is independent of the optimal x. If we plug in the parameters, we get that $y = \frac{1}{4}\cdot\frac{20}{1} = 5$. This is the formula for the price offer curve!

Plotting this price offer curve (in black) and the optimal bundles for $p_x = 1, 3, and 5$ below confirms that each of the tangency points lies on the price offer curve. This creates a locus on all optimal bundles.

3 Cross-Price

As the name implies, here we are interested in how the other price affects our optimal consumption of x and y. We want to fix own-price and income while allowing the price of the other good to vary. This asks: how does the quantity of good i change as the price of good j changes, holding all else constant? Interestingly, this tells us if the goods are compliments or substitutes. This offer curve is called the price offer curve.

[1] I think this is also called the price consumption locus, but I am not sure. I think it's easiest to just call these different offer curves.



3.1 Perfect Compliments

Recall that our demand functions for perfect compliments taking the form $u(x, y) = min\beta x, \alpha y$ can be written in the following general case:

$$x^*(p, I) = \frac{\alpha I}{\alpha p_x + \beta p_y}$$

$$y^*(p, I) = \frac{\beta I}{\alpha p_x + \beta p_y}$$

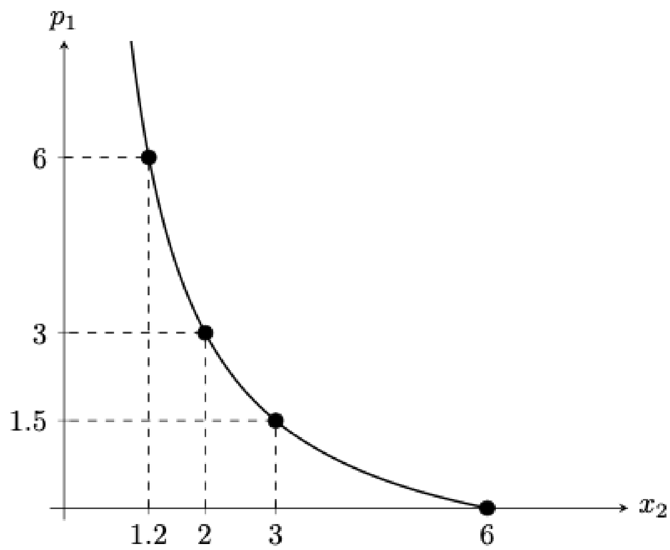
Let's consider the following parameters: $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, p_x = 3, p_y = 2, I = 12$. Let's just plug these in to find the demand functions of the other price for the above parameters:

$$x^*(p, I) = \frac{\frac{1}{3} \cdot 12}{\frac{1}{3} \cdot 3 + \frac{1}{4} \cdot p_y} = \frac{16}{4 + p_y}$$

$$y^*(p, I) = \frac{\frac{1}{4} \cdot 12}{\frac{1}{3} \cdot p_x + \frac{1}{4} \cdot 2} = \frac{18}{2p_x + 3}$$

Notice how in the general case, since prices are in the denominator, when the price of good i rises, the consumer will consume less good j. This tells us that the good are compliments (surprising considering the name, huh?). In any case, we will need to plot the demand curves for both goods.

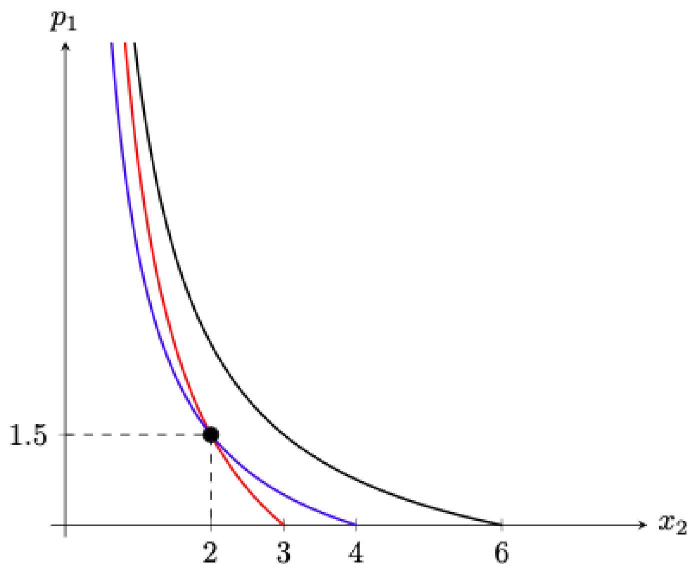
Let's start with y as a function of p_x. Again, we start by solving for p_x which should give us the inverse plot $p_x(y, p_y, I)$:



$$y = \frac{\frac{1}{4}I}{\frac{1}{3} + \frac{1}{4}p_y} = \frac{3I}{4p_x + 3p_y}$$
$$\therefore 4p_x + 3p_y = \frac{3I}{y}$$
$$p_x = \frac{1}{4}(\frac{3I}{y} - 3p_y)$$
$$p_x(y, p_y, I) = \frac{3I}{4y} - \frac{3}{4}p_y$$
$$\therefore p_x(y, 2, 12) = \frac{9}{y} - \frac{3}{2}$$

I will plot this and comment on a few key features. I will also highlight a few important price levels: $p_x = 1.5$, $p_x = 3$, $p_x = 6$, and $p_x = 0$. This gives us the graph below. Let's note a few key features of this demand curve compared to the Cobb-Douglas demand curve. Firstly, as always, we have price graphed on the vertical axis. Secondly, although we never really consider the case $p_x = 0$ (since when are prices free in the real world?) we do see that the curve can actually touch the x-axis! This is unlike Cobb-Douglas demand curves, which never touch the axes. This is kind of cool, because we can imagine a situation in which the price of the compliment (other good i) is so expensive, the consumer will just consume zero of good j. Right then, the lesson is to check the intercepts!

Now, let's change the price of good y and have it increase to $p_y = 4$. By the law of demand, we expect y to decrease. Now, a more interesting question is how this affects the relationship between y and p_x. We see from the demand curve $p_x(y, p_y, I)$ that a change in p_y only really affects the intercept of the graph – not the slope term. Likewise, if I decreases to, say, 8, we will have a flatter



slope, demand will fall, and the relationship between y and p_x will increase. To see this, we will note the new parameter changes below:

$$y(p_x, 4, 12) = \frac{3 \cdot 12}{4p_x + 3 \cdot 4} = \frac{9}{p_x + 3}$$

$$p_x(y, 4, 12) = \frac{3 \cdot 12}{4y} - \frac{3}{4} \cdot 4 = \frac{9}{y} - 3$$

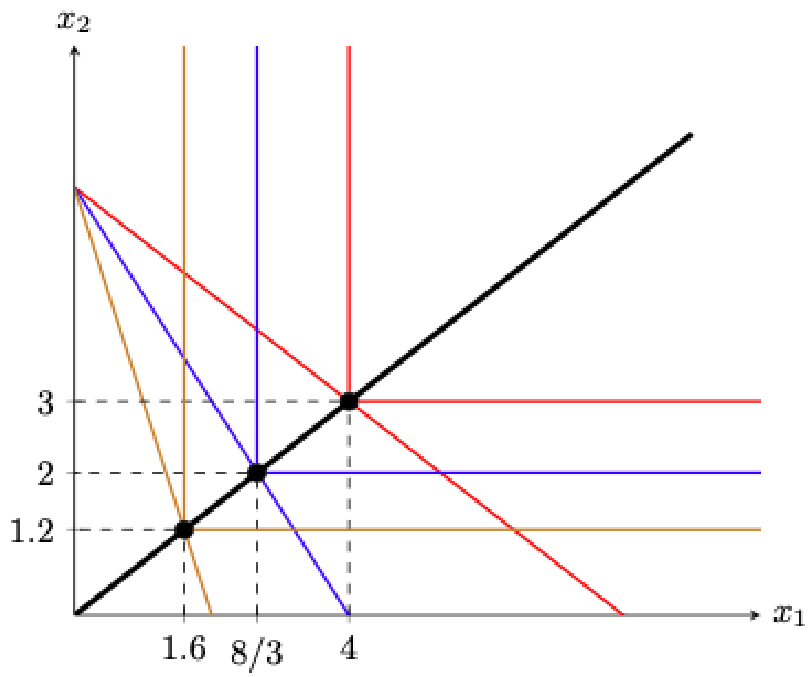
$$y(p_x, 2, 8) = \frac{3 \cdot 8}{4p_x + 3 \cdot 2} = \frac{12}{2p_x + 3}$$

$$p_x(y, 2, 8) = \frac{3 \cdot 8}{4y} - \frac{3}{4} \cdot 2 = \frac{6}{y} - \frac{3}{2}$$

From here, I will plot the change in p_y in red and the change in I in blue. This gives us the following diagram.

Let's finally do the price offer curve. Here, we will do the p_x price offer curve. Let's use the same initial parameter values above: $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, p_x = 3, p_y = 2, I = 12$. Now, our price offer curve is the locus of points satisfying the following conditions:

$$(x(p_x, p_y, I), y(p_x, p_y, I)) = (\frac{\frac{1}{3}I}{\frac{1}{3}p_x + \frac{1}{4}p_y}, \frac{\frac{1}{4}I}{\frac{1}{3}p_x + \frac{1}{4}p_y}) = (\frac{4I}{4p_x + 3p_y}, \frac{3I}{4p_x + 3p_y})$$



$$\therefore (x(p_x, 2, 12), y(p_x, 2, 12)) = (\frac{48}{4p_x + 6}, \frac{36}{4p_x + 6}) = (\frac{24}{2p_x + 3}, \frac{18}{2p_x + 3})$$

This one is a bit harder to visualize than the Cobb-Douglas locus. So, let's think about the formula for the price offer curve. We first need the tangency condition, which for perfect complements occurs at the kinked indifference curve point.

$$y = \frac{\beta}{\alpha}x$$

Notice, then, that this does not have prices (specifically, p_x) in it! We are done! This is exactly the formula for the price offer curve:

$$y(x, p_y, I) = \frac{\beta}{\alpha}x = \frac{1/4}{1/3}x = \frac{3}{4}x$$

Now, we must plot the offer curve as well as the optimal bundles for $p_x = 1.5$, $p_x = 3$, and $p_x = 6$. And now we are done!

3.2 Perfect Substitutes

Let's finally consider a perfect substitutes example: $u(x,y) = \beta x + \alpha y$. We know that their demand function should look like:

$$(x,y) = \begin{cases} (\frac{I}{p_x}, 0) & if \frac{MU_x}{p_x} > \frac{MU_y}{p_y}, \\ (0, \frac{I}{p_y}) & if \frac{MU_x}{p_x} < \frac{MU_y}{p_y}, \\ BudgetLine & if \frac{MU_x}{p_x} = \frac{MU_y}{p_y}. \end{cases}$$

This means that we can write the demand for y as a function of p_x as:

$$y(p_x, p_y, I) = \begin{cases} (\frac{I}{p_x}, 0) & if \frac{MU_x}{p_x} < \frac{MU_y}{p_y}, \\ (0, \frac{I}{p_y}) & if \frac{MU_x}{p_x} > \frac{MU_y}{p_y}, \\ BudgetLine & if \frac{MU_x}{p_x} = \frac{MU_y}{p_y}. \end{cases}$$

Let's set $MU_x = 2$ and $MU_y = 1$. We will consider the following parameter values:

Parameters: (plot in blue)

$$p_x = 1, p_y = 2, I = 10$$
$$\underline{Demand} : y(p_x, 2, 10)$$
$$\begin{cases} 5 & if p_x > 1, \\ \in [0,5] & if p_x = 1, \\ 0 & if p_x < 1. \end{cases}$$

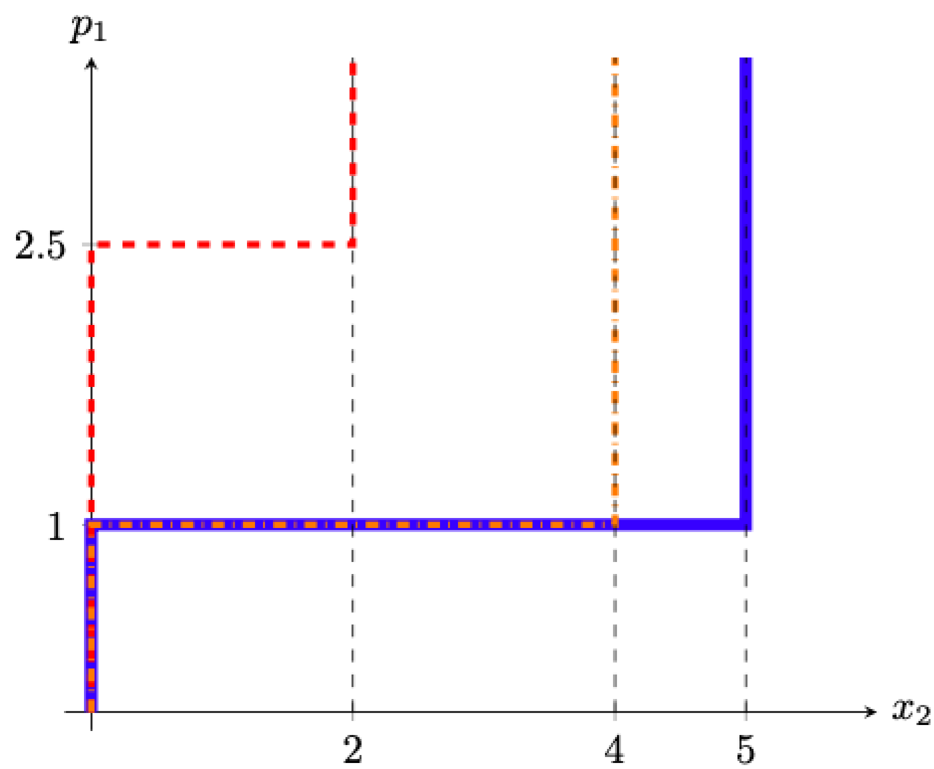
Parameters: (plot in red)

$$p_x = 1, p_y = 5, I = 10$$
$$\underline{Demand} : y(p_x, 5, 10)$$
$$\begin{cases} 2 & if p_x > 2.5, \\ \in [0,2] & if p_x = 2.5, \\ 0 & if p_x < 2.5. \end{cases}$$

Parameters: (plot in orange)

$$p_x = 1, p_y = 2, I = 8$$
$$\underline{Demand} : y(p_x, 2, 8)$$
$$\begin{cases} 8 & if p_x < 4, \\ \in [0,8] & if p_x = 4, \\ 0 & if p_x > 4. \end{cases}$$

We (I) plot these graphs below. The colors are somewhat hard to see, thanks to this stupid Latex package, but you should try drawing this yourself in the meantime. Notice how a change in I does not affect the cutoff value (where there is a straight line) but instead only affects how much good y is purchased. However, changes in p_x do change both how much y is purchased as well as the cutoff.



p_1
2.5
1
2
4
5
x_2

4 Income

I am really tired at this point, but the theme of this section is similar: we want to isolate the consumer's income and plot it against the quantity of good x or y they demand. This asks: how does the quantity of good i change as income changes, holding all else constant? We can call this graph an Engel curve. The offer curve here is called an income offer curve.

4.1 Cobb-Douglas

Returning to our good friends Cobb and Douglas, we will use the same parameters as last time: $\alpha = .75$, $\beta = .25$, $p_x = 3$, $p_y = 1$, and $I = 20$. Our Engel curves are:

$$x(I, p_x, p_y) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{p_x}$$
$$\therefore x(I, 3, 1) = \frac{I}{4}$$

$$y(I, p_x, p_y) = \frac{\alpha}{\alpha + \beta} \cdot \frac{I}{p_y}$$
$$\therefore y(I, 3, 1) = \frac{I}{4}$$

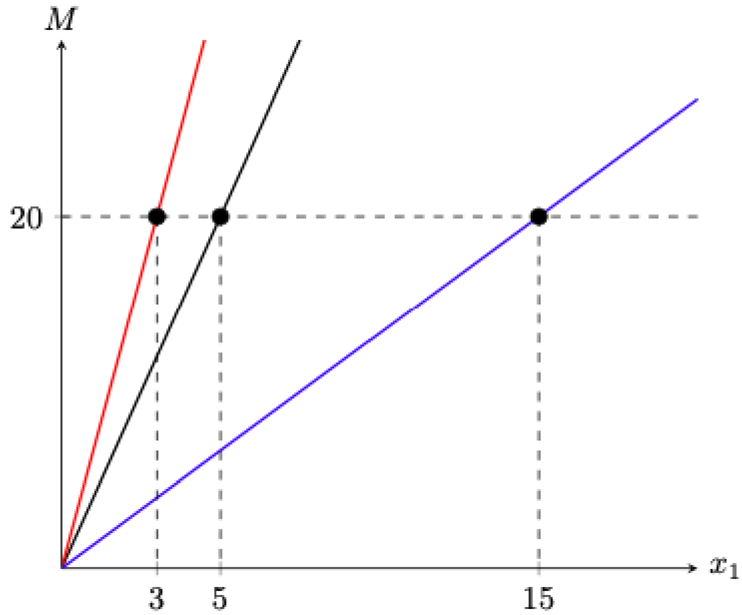
As per usual, we take the inverse of our demand function to plot it. This time, we are plotting:

$$\therefore I(x, 3, 1) = 4x$$
$$\therefore I(y, 3, 1) = 4y$$

I plot the Engel curves below, as well as for $x(I, 5, 1)$ (in red) and $x(I, 1, 2)$ (in blue). At this point, just solve it yourself. You are more than capable and should definitely be able to before the second midterm.

To no one's surprise, Cobb-Douglas preferences give us normal goods! As income increases, we demand more of the goods. We also know that a property of Cobb-Douglas is that the share of income spent on each good stays constant. Since we hold prices fixed, this means that an increase in income will lead to a proportional increase in quantity demanded. In other words, our Engel curves are linear. This is what we demonstrated above.

This property is called **homotheticity**. The lecture slides cover this. Remind me to add this later.



5 Compensated (Hicksian) Demand

5.1 Overview

I am going to rush through this for sake of time. The Hicksian demand is the demand we find when we want to minimize expenditures subject to utility. What does this mean? It means we want to minimize the amount we have to spend in order to reach some fixed utility, U.

The setup is almost the exact same as for Marshallian demand.

1. Find the tangency condition. Set MRS $= \frac{p_x}{p_y}$
2. Solve for either x or y
3. Plug value for x or y into the utility function, $u(x, y)$.
4. Solve for x^c or y^c

5.2 Cobb-Douglas Example

Say we have the following utility function:

$$u = x^2 y$$

Let's just find the compensated demand:

$$MRS = \frac{p_x}{p_y}$$

$$\frac{2y}{x} = \frac{p_x}{p_y}$$

$$y = \frac{xp_x}{2p_y}$$

So now that we have the tangency condition, we have to plug it into our new constraint: the utility function. We know that $u(x, y) = x^2y$, so we will have to replace y.

$$u = (x^2 \frac{xp_x}{2p_y})$$

$$u = (\frac{p_x}{2p_y})x^3$$

$$x^3 = (\frac{2p_y u}{p_x})$$

$$x^c = \sqrt[3]{\frac{2p_y u}{p_x}}$$

And now we have the compensated demand for good x. To find it for y, we do the same thing as before from the tangency condition. I will leave this as an exercise for you, but you should get:

$$y^c = \sqrt[3]{(\frac{2p_y}{p_x})^2 u}$$

5.3 Perfect Compliments Example

Say we have the following utility function:

$$u(x, y) = min3x, y$$

We know that we cannot differentiate this function to find the tangency condition, so what do we do instead? Well, this is where we have to use our economic intuition to get our compensated demand. We know that we consume at the kink point. In other words, our optimal bundle occurs when we equate the inner terms. If $u = min\frac{1}{\alpha}x, \frac{1}{\beta}y$, we must consume where $\frac{1}{\alpha}x = \frac{1}{\beta}y$, otherwise we are inefficiently consuming goods that yield no marginal utility (for further discussion, refer to my optimal choice notes).

So we have that, written out:

$$u = \frac{1}{\alpha}x = \frac{1}{\beta}y$$

This means that our utility is just equal to both terms! Turns out that perfect compliments are much easier in terms of finding compensated demand. Let's refer back to our original problem.

$$u(x, y) = min3x, y$$

$$u = 3x = y$$

Here, we know that we can just solve in terms of x and y.

$$y^c = u$$

and

$$3x = u$$
$$x^c = \frac{u}{3}$$

There we have it! We have just solved for the compensated demand.

5.4 Perfect Substitutes Example

Say we have the following utility function:

$$u(x, y) = 5x + 2y$$

We know that we cannot differentiate this function and solve for a tangency, because these preferences are perfect substitutes. The indifference curves are downward sloping lines, so they will not have points of tangency with the budget constraint. In the case of compensated demand, we are trying to minimize expenditures subject to utility, so we are trying to match the perfect budget constraint subject to utility. This means that we are going to have another corner solution!

How do we find corner solutions for perfect substitutes? We compare marginal utility per dollar! So for this case, we set up the relationship between marginal utilities per dollar.

$$\frac{5}{p_x} = \frac{2}{p_y}$$

So we now have a relationship where we know that the marginal utilities are equal. If $\frac{MUx}{p_x} > \frac{MUy}{p_y}$, we will have a corner solution at the x intercept. But normally, for ordinary demand, this intercept is given by $\frac{I}{p_x}$. Is this still the case for Hicksian (compensated) demand? How has our constraint changed?

Since our constraint is no longer reliant on income, we then have an intercept at $x = \frac{u}{p_x}$. This leads to a very staggering result: if $\frac{MUx}{p_x} > \frac{MUy}{p_y}$, then $x^x = \frac{u}{p_x}$.

Let's solve the given problem with $p_x = 10$. We know that

$$u(x, y) = 5x + 2y$$

And that $p_x = 10$. We can the marginal utilities per dollar to be:

$$\frac{5}{10} = \frac{2}{p_y}$$

We can now see that marginal utility per dollar for good x is just $\frac{1}{2}$. This means that as long as p_y is less than 4, we will only consume good y. Likewise, if $p_y > 4$, we will only consume good x. We then have the following compensated demand:

$$x^c = \begin{cases} \dfrac{u}{p_x} & if p_y > 4, \\ \in [0, \dfrac{u}{p_x}] & if p_y = 4, \\ 0 & if p_y < 4. \end{cases}$$

Et voila. These are the cases for x^c.

5.5 Quasi-linear Example

These functions are going to be kind of brutal. Well, they can be. Let's do an example of a simple quasi-linear.

$$u(x, y) = ln(x) + y$$

$$\frac{\frac{1}{x}}{1} = \frac{p_x}{p_y}$$

$$x = \frac{p_y}{p_x}$$

Throw this into our utility function:

$$u = ln(\frac{p_y}{p_x}) + y$$

Rearranging to solve for y:

$$y^c = u - ln(\frac{p_y}{p_x})$$

However, like with all quasi-linear functions, the value of y could be negative. So we have to see whether y is positive. Setting $y = 0$ we can solve for the edge case:

$$u = ln(\frac{p_y}{p_x})$$

So we conclude that the value of utility must be greater than the natural log of the price ratio in order for the consumer to have $y^c > 0$.

Overview

- why use CV, EV, CS?
  - ↳ preciser measure of how much better off a consumer may be.
  - ↳ recall: utility tells us how consumers rank bundles. Meaningless otherwise.

CS

- How much you are willing to pay above price.

  new world - old world

EV

- How much money to get you to same utility as in new world while keeping prices same as old world.

CV

- How much money to get you to same utility as in the old world, while keeping prices same as the new world.

	old utility	new utility
old prices	old world	EV world
new prices	CV world	New world

There is no SE → only IE.

→ Means prices will be the same

→ adjust income to get us to new/old world Utility.

The EV & CV will not be the same. Why?

Different reference points: [prices]

If EV < 0 or CV > 0 : welfare falls

EV → Starting at old world, I must take money away from you [EV < 0] to make you indifferent to new world.

CV → Starting at new world I must give you money [CV > 0] to make you indifferent to old world.

If EV > 0 or CV < 0 : welfare rises

EV → Starting @ old world, to make you indifferent to new world, I need to give you money.

→ Starting @ new world I need to take money away to make you indif. to old world.

Area of triangle: $\frac{1}{2}(b)(h)$

Area of trapezoid: $\frac{1}{2}(a+b)(h)$

1) ex) $Q = 400 - 20P$

$P = 15 \rightarrow P = 8$

ΔCS?

Sol: $20P = 400 - Q$

$P = 20 - \frac{1}{20}Q$

@15

$Q = 400 - 20(15)$

$Q = 400 - 300$

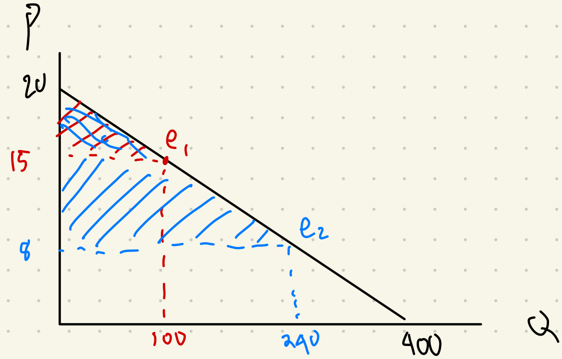
$= 100$

@ 8

$Q = 400 - 20(8)$

$= 400 - 160$

$= 240$



$CS_1: \frac{1}{2}(20-15)(100-0) = 250$

$CS_2: \frac{1}{2}(20-8)(240) = 1440$

$\Delta CS = 1440 - 250 = 1190$

old CS: $\frac{1}{2}(20-15)(100-0)$

2 C-D

$U = x_1^3 x_2^2$ $\quad P_1 = 8, \; P_2 = 2, \; I = 120$

P_1 falls to 4

1) MRS: $\frac{3x_1^2 x_2^2}{2x_2 x_1^3} \Rightarrow \underbrace{3x_1^{2-3} \; y^{2-1}}_{2} = \frac{3y}{2x}$

$$\frac{3x_2}{2x_1} = \frac{P_1}{P_2} \Rightarrow x_2 = \frac{2P_1}{3P_2} x_1$$

$$\therefore P_1 x_1 + P_2\left(\frac{2P_1}{3P_2} x_1\right) = I$$

$$P_1 x_1 \left(1 + \frac{2}{3}\right) = I$$

$$x_1 = \frac{3I}{5P_1} \qquad \left[\frac{2\cancel{P_1}}{3P_2}\right]\left[\frac{\cancel{3}I}{5\cancel{P_1}}\right]$$

$$x_2 = \frac{2I}{5P_2}$$

Before

$$x_1(8, 2, 120) = \frac{3 \cdot 120}{5 \cdot 8} = 9$$

$$x_2(8, 2, 120) = \frac{2 \cdot 120}{5 \cdot 2} = 24$$

After

$$x_1(4, 2, 120) = \frac{3 \cdot 120}{5 \cdot 4} = 18$$

$$x_2(4, 2, 120) = \frac{2 \cdot 120}{5 \cdot 2} = 24$$

EV: old prices with I^E

prices must be the same, so x^e...

$$(x_1^e, x_2^e) = \left(\frac{3I^e}{5P_2^0}, \frac{2I^e}{5P_2^0}\right) = \left(\frac{3I^e}{5 \cdot 8}, \frac{2I^e}{5 \cdot 2}\right) = \frac{3I^e}{40}, \frac{2I^e}{10}$$

new utility: $u' = x_1'^{3} x_2'^{2}$
$= 18^3 \cdot 24^2$

Solve w/ I^e : $u(x_1^e, x_2^e) = u'$

$$\left(\frac{3 I^e}{40}\right)^3 \left(\frac{I^e}{5}\right)^2 = 18^3 \cdot 24^2$$

$$I^{e5} \left(\frac{3}{40}\right)^3 \left(\frac{1}{5}\right)^2 = 18^3 \cdot 24^2$$

$$I^{e5} = 18^3 \cdot 24^2 \cdot \left(\frac{3}{40}\right)^{-3} \cdot \left(\frac{1}{5}\right)^{-2}$$

$$I^{e5} = \left(\frac{18 \cdot 40}{3}\right)^3 \cdot (24 \cdot 5)^2$$

raise to $\frac{1}{5}$

$$I^e = (240)^{3/5} \cdot (120)^{2/5}$$

$$I^e = 120 \cdot 2^{3/5}$$

$$I^e \approx 181.86$$

EV: $I^e - I^0$
$\approx 181.86 - 120$
≈ 61.86

Price drop has same effect as giving \$61.86

CV

new prices $ I^c

$$4x_1 + 2x_2 = I^c$$

$$(x_1^c, x_2^c) = \left(\frac{3I^c}{5p_1'}, \frac{2I^c}{5p_2'}\right) = \left(\frac{3I^c}{5\cdot 4}, \frac{2I^c}{5\cdot 2}\right) = \left(\frac{3I^c}{20}, \frac{2I^c}{10}\right)$$

$$u^0 = (x_1')^3 (x_2')^2$$

$$= 9^3 \cdot 24^2$$

$$u(x_1^c, x_2^c) = u^0$$

$$\left(\frac{3I^c}{20}\right)^3 \left(\frac{I^c}{5}\right)^2 = 9^3 \cdot 24^2$$

$$I^{c^5} \left(\frac{3}{20}\right)^3 \left(\frac{1}{5}\right)^2 = 9^3 \cdot 24^2$$

$$I^{c^5} = 9^3 \cdot 24^2 \cdot \left(\frac{3}{20}\right)^{-3} \cdot \left(\frac{1}{5}\right)^{-2}$$

$$I^{c^5} = \left(\frac{9 \cdot 20}{3}\right)^3 \cdot (24 \cdot 5)^2$$

$$I^c = \left[60\left(\tfrac{1}{3}\right)\right]^{3/5} \cdot [120]^{2/5}$$

$$\approx 79.17$$

CV: $79.17 - 120 = -40.83$ } Take money away to "compensate" for price drop.

3 Quasi-linear

$U = 4\sqrt{x_1} + 2x_2$ $\qquad P_1 = 2$, $I = 10$

$P_2 = 2$, changes to 4

demand: $MRS = \dfrac{4 \cdot \frac{1}{2} x_1^{-1/2}}{2} = \dfrac{P_1}{P_2}$

$$x_1^{-1/2} = \frac{P_1}{P_2}$$

$$x_1 = \left(\frac{P_2}{P_1}\right)^2$$

$10 = P_1 x + P_2 y$

$10 = P_1\left[\frac{P_2}{P_1}\right]^2 + P_2 y$

$10 = \frac{P_2^2}{P_1} + P_2 y$

$y P_2 = 10 - \frac{P_2^2}{P_1}$

$y = \frac{10}{P_2} - \frac{\frac{P_2^2}{P_1}}{P_2}$

$y = \frac{10}{P_2} - \frac{P_2}{P_1}$

$$x_2 = \frac{1}{P_2}\left[I - P_1 x_1(P, I)\right]$$

$$= \frac{I}{P_2} - \frac{P_1}{P_2}\left(\frac{P_2}{P_1}\right)^2$$

$$= \frac{I}{P_2} - \frac{P_2}{P_1}$$

Before

$x_1(2, 2, 10) = \left(\frac{2}{2}\right) = 1^2 = 1$

$x(2, 2, 10) = \frac{10}{2} - \frac{2}{2} = 4$

After

$x_1(2, 4, 10) = 2^2 = 4$

$x_2(2, 4, 10) = \frac{10}{4} - \frac{4}{2} = \frac{1}{2}$

EV

$$2x_1 + 2x_2 = I^e$$

$$(x_1^e, x_2^e) = \left[\left(\frac{P_2^0}{P_1^0}\right)^2, \frac{I^e}{P_1^0} - \frac{P_2^0}{P_1^0}\right] = \left(\left(\frac{2}{2}\right)^2, \frac{I^e}{2} - \frac{2}{2}\right)$$

$$= 1, \frac{I^c}{2} - 1$$

$$u' = 4\sqrt{x_1'} + 2x_2'$$
$$= 4\sqrt{4} + 2\left(\frac{1}{2}\right)$$
$$= 9$$

$$u(x_1^e, x_2^e) = u'$$

$$4\sqrt{1} + 2\left[\frac{I^e}{2} - 1\right] = 9$$

$$4 + I^c - 2 = 9$$

$$I^c = 7$$

$$CV = I^e - I^0$$

$$= 7 - 10$$
$$= -3$$

] take \$3 !!

CV

$$2x_1 + 4x_2 = I^c$$

new prices old util

$$x_1^c, x_2^c = \left[\left(\frac{p_2'}{p_1'}\right)^2, \frac{I^c}{p_2'} - \frac{p_2'}{p_1'}\right) = \left[\left(\frac{4}{2}\right)^2, \frac{I^c}{4} - \frac{4}{2}\right]$$

$$\left[4, \frac{I^c}{4} - 2\right]$$

$$u^0 = 4\sqrt{x_1^0} + 2x_2^0$$

$$= 4\sqrt{1} + 2 \cdot 4$$

$$= 12$$

$$u(x_1^c, x_2^c) = u^0$$

$$4\sqrt{4} + 2\left(\frac{I^c}{4} - 2\right) = 12$$

$$8 + \frac{I^c}{2} - 4 = 12$$

$$I^c = 16$$

$$CV = I^c - I^0$$

$$= 16 - 10$$

$$= 6$$] compensate by giving $6.

numeraire = linear term

Bonus

Numeraire CV & EV [Quasi-linear cont.]

P_1 changes to 4

$$x_1(4,2,10) = \left(\frac{2}{4}\right)^2 = \left(\frac{1}{2}\right)^2 = .25$$

$$x_2(4,2,10) = \frac{10}{2} - \frac{2}{4} = 5 - \frac{1}{2} = 4.5$$

EV: $2x_1 + 2x_2 = I^e$

$$x_1^e, x_2^e = \left(\left(\frac{p_2^0}{p_1^0}\right)^2, \frac{I^e}{p_2^0} - \frac{p_2^0}{p_1^0}\right) = \left(\left(\frac{2}{2}\right)^2, \frac{I^e}{2} - \frac{2}{2}\right)$$

$$= \left[1, \frac{I^e}{2} - 1\right]$$

$$u' = 4\sqrt{x_1'} + 2x_2'$$

$$= 4\sqrt{\tfrac{1}{4}} + 2(4.5)$$

$$= 4\left(\tfrac{1}{2}\right) + 9$$

$$u' = 11$$

$$u(x_1^e, x_2^e) = u'$$

$$4\sqrt{1} + 2\left(\frac{I^e}{2} - 1\right) = 11$$

$$4 + I^e - 2 = 11$$

$$I^e = 9$$

$$EV = u^e - u'$$
$$= 9 - 10$$
$$= -1$$

CV

$$4x_1 + 2x_2 = I^c$$

$$(x_1^c, x_2^c) = \left(\left(\frac{p_2^1}{p_1^1}\right)^2, \frac{I^c}{p_2^1} - \frac{p_2^1}{p_1^1} \right) = \left[\left(\frac{2}{4}\right)^2, \frac{I^c}{2} - \frac{2}{4} \right]$$

$$= \left(\frac{1}{4}, \frac{I^c}{2} - \frac{1}{2} \right]$$

$$u^0 = 4\sqrt{1} + 2 \cdot 4 = 12$$

$$u(x_1^c, x_2^c) = u^0$$

$$4\sqrt{\frac{1}{4}} + 2\left[\frac{I^c}{2} - \frac{1}{2} \right] = 12$$

$$2 + I^c - 1 = 12$$

$$I^c = 11$$

$$CV = I^c - I^0$$

$$= 11 - 10$$

$$= -1$$

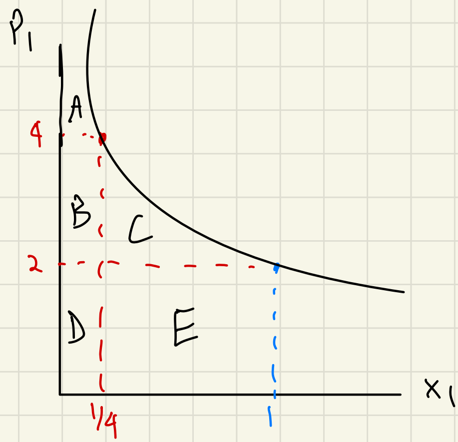
CS

$X_1 = \left(\frac{P_2}{P_1}\right)^2$ invert $\Rightarrow$ $P_1 = \frac{P_2}{\sqrt{x_1}}$

go from $P_1 = 2$ to $P_1 = 4$

X changes from 1 to $\frac{1}{4}$

$$\int_{\frac{1}{4}}^{1} P_1(x_1; 2, 10) = \int_{\frac{1}{4}}^{1} \frac{2}{\sqrt{x_1}} = \int_{1/4}^{1} 2x_1^{-\frac{1}{2}}$$



$$= \left[2 \cdot 2x_1^{\frac{1}{2}}\right]_{1/4}^{1}$$
$$= 4\sqrt{1} - 4\sqrt{\tfrac{1}{4}}$$
$$= 4 - 4\left(\tfrac{1}{2}\right)$$
$$= 4 - 2$$
$$= 2$$

Why not 1?

Δ Gross CS = C+E
- old gross = A+B+C+D+E
- new gross = A+B+D

Δ Net CS = B+C
- old net = A+B+C
- new net = A

$B = (4-2)(\frac{1}{4}-0) = \frac{1}{2}$

$E = (2-0)(1-\frac{1}{4}) = 1.5$

$$\Delta CS_{net} = \Delta CS_{gross} + B - E$$
$$= 2 + \tfrac{1}{2} - 1.5$$
$$= 1$$

4 Perfect Comps

$U(x_1, x_2) = \min(3x_1, 4x_2)$

$P_1 = 1,\ P_2 = 2,\ m = 30$

$P_1 \uparrow$ to 3.

$3x_1 = 4x_2$

$x_2 = \frac{3}{4}x_1$

$\therefore\ P_1 x_1 + P_2\left(\frac{3}{4}x_1\right) = I$

$x_1(4P_1 + 3P_2) = 4I$

$x_1 = \dfrac{4I}{4P_1 + 3P_2}$

$x_2 = \dfrac{3I}{4P_1 + 3P_2}$

$4P_1x_1 + 3P_2x_1 = 4m$

$x_1[4p_1 + 3P_2] = 4m$

$x = \dfrac{4m}{4p_1 + 3P_2}$

before

$x_1 = \dfrac{4(30)}{4(1) + 3(2)} = 12$

$x_2 = \dfrac{3 \cdot 30}{4(1) + 3(2)} = 9$

after

$x_1 = \dfrac{4 \cdot 30}{4(3) + 3(2)} = \dfrac{20}{3}$

$x_2 = \dfrac{3 \cdot 30}{4(3) + 3(2)} = 5$

EV

$$x_1 + 2x_2 = I^e$$

$$x_1^e, x_2^e = \left[\frac{4I^e}{4p_1^0 + 3p_2^0}, \frac{3I^e}{4p_1^0 + 3p_2^0} \right)$$

$$= \frac{4I^e}{4 \cdot 1 + 3 \cdot 2}, \frac{3I^e}{4 \cdot 1 + 3 \cdot 2}$$

$$= \frac{4I^e}{10}, \frac{3I^e}{10}$$

$$u^1 = \min\left(3 \cdot \frac{20}{30}, 4 \cdot 5\right)$$

$$= (20, 20) \quad = 20$$

$$\min\left\{3 \cdot \frac{4I^e}{10}, 4 \cdot \frac{3I^e}{10}\right\} = 20$$

$$\frac{12 I^e}{10} = 20$$

$$I^e = \frac{200}{12} = \frac{100}{6} \approx 16.67$$

$$EV = I^e - I^0$$

$$= 16.67 - 30$$

$$= -13.33$$

5 Perf. Subs

$u = 2x_1 + x_2$

$P_1 = 3,\ P_2 = 5,\ I = 15$

(1) $P_2 \downarrow$ to 3

(2) $P_2 \downarrow$ to 1

$MRS = \frac{2}{1} = 2$

$$x_1(P, I) = \begin{cases} 0 & 2 < \frac{P_1}{P_2} \\ \frac{I}{P_1} & 2 > \frac{P_1}{P_2} \\ \in \left[0, \frac{I}{P_1}\right] & 2 = \frac{P_1}{P_1} \end{cases}$$

$$x_2(P, I) = \begin{cases} 0 & 2 > \frac{P_1}{P_2} \\ \frac{I}{P_2} & 2 < \frac{P_1}{P_2} \\ \in \left[0, \frac{I}{P_2}\right] & 2 = \frac{P_1}{P_2} \end{cases}$$

Before	(1)	(2)
$\frac{P_1}{P_2} = .6$	$\frac{P_1}{P_2} = 1$	$\frac{P_1}{P_2} = 3$
$x_1 = 5$	$x_1 = 5$	$x_1 = 0$
$x_2 = 0$	$x_2 = 0$	$x_2 = 15$

$3x_1 + 3x_2 = I^c$ (1)

$3x_1 + x_2 = I^e$ (2)

(1)

$x_1^c, x_2^c = \frac{I^c}{p_1'}, 0$

$= \frac{I_c}{3}, 0$

(2)

$x_1^c, x_2^c = \left(0, \frac{I^c}{p_2'}\right)$

$= 0, I^c$

$u(x_1^e, x_2^e) = u^0$ so old util = 10

$u^0 = 2x_1^0 + x_2^0$
$= 2(5) + 0$
$= 10$

2. $\frac{I^c}{3} + 0 = 10$

$I^c = 15$

CV: $I^c - I^0$

$= 15 - 15$
$= 0$

didn't change bundles

$u(x_1^c, x_2^c) = u^0$
$2(0) + I^c = 10$
$I^c = 10$

CV: $I^c - I^0$
$= 10 - 15$
$= -5$